

EIGENVALUE ASYMPTOTICS FOR AN ELASTIC STRIP AND AN ELASTIC PLATE WITH A CRACK

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ABSTRACT. We consider the elasticity operator with zero Poisson's ratio on an infinite strip and an infinite plate with a horizontal crack. We prove an asymptotic formula for the distance of the embedded eigenvalues to some spectral threshold of the operator as the crack becomes small.

1. INTRODUCTION

In the present article we consider an elastic strip and an elastic plate with a horizontal crack. We are interested in the existence of trapped modes and their asymptotic behaviour as the crack shrinks to a point. Mathematically a trapped mode corresponds to an (embedded) eigenvalue of the elasticity operator acting on functions which satisfy traction-free boundary conditions. We assume that the elastic material is homogeneous and isotropic having zero Poisson's coefficient. We generalise previous results obtained in [1], where the existence of at least two embedded eigenvalues for the infinite strip with a horizontal crack has been proved. Moreover, for an elastic plate with crack it was shown that there exist infinitely many eigenvalues and a one-sided asymptotic estimate for the distance of the eigenvalues to some spectral threshold has been given. We generalise this estimate and prove an asymptotic formula. Instead of the variational ansatz, which has been used in [1], we use a boundary integral method based on the corresponding Dirichlet-to-Neumann mapping.

As in [1] we want to take advantage of the symmetry of the domain, which allows us to consider an equivalent mixed problem. An additional symmetry induced by the choice of Poisson's ratio allows us to prove the existence of exactly two discrete eigenvalues of some symmetric part of the operator. For non-zero Poisson's ratio this last symmetry decomposition fails and the eigenvalues will in general turn into resonances. Although, the method is suitable to treat this problem, this case will not be considered in the present work. The symmetry decomposition of the elasticity operator for vanishing Poisson's ratio goes back to [2], where the existence of embedded eigenvalues was shown for the elastic semistrip. These results have been generalised to the case of an elastic strip or plate with zero Poisson's ratio where material properties are perturbed [3, 4], where a hole is cut out [5], or strips and plates having a crack [1]. In [3, 4] an asymptotic formula for the convergence of the eigenvalues to some spectral threshold was proved. The eigenvalue expansion is constructed via a Birman-Schwinger analysis of the original operator. In contrast to the perturbation of material properties, in the case of a shrinking

crack the domain of the elasticity operator¹ does not remain unchanged. Hence, the Birman-Schwinger cannot be applied. Moreover, the variational approach which was used in [1] is not suitable to obtain an asymptotic formula for the eigenvalues. In the present article we want to use a boundary integral approach for the proof of the asymptotic formula. This method allows us to reformulate the singular perturbation of the original operator into an additive perturbation of the Dirichlet-to-Neumann operator. Then the asymptotic formula for the eigenvalues follows by a Birman-Schwinger analysis of the corresponding operator acting on the boundary. The method was outlined in the simple case of the Dirichlet Laplacian with Neumann window in [6].

For bounded domains a similar approach has been employed in [7] to prove an asymptotic formula for the eigenvalues of the Laplacian acting on a domain with a small inclusion, cf. also [8] for the treatment of an elastic crack problem.

In addition in the case of an infinite elastic waveguide, there are various numerical approaches, see e.g. [9, 10, 11, 12, 13, 14]. However, to our knowledge the articles [1, 2, 3, 4, 5] contain the only analytical results for a perturbed elastic strip or plate.

2. STATEMENT OF THE PROBLEM

Let $I := (-\pi/2, \pi/2)$. We denote by $\Omega^{(d)} := \mathbb{R}^{d-1} \times I$ an elastic medium in dimension $d \in \{2, 3\}$ and consider cracks of the form $\bar{\Gamma} \times \{0\}$, where $\Gamma \subseteq \mathbb{R}^{d-1}$ is a bounded open set. As there is no risk of confusion we identify $\bar{\Gamma} \times \{0\}$ with $\bar{\Gamma}$. In what follows we denote by $u : \Omega^{(d)} \setminus \bar{\Gamma} \rightarrow \mathbb{C}^d$ the displacement field of the elastic material. Its strain is given by the matrix

$$E(u) := \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j=1,\dots,d} \quad (2.1)$$

and for the stress we have

$$\Sigma(u) := 2\mu E(u) + \lambda \operatorname{div}(u) I_d, \quad (2.2)$$

where I_d is the d -dimensional unit matrix and λ and μ are the Lamé coefficients. These constants depend on Poisson's ratio and Young's modulus of the material and satisfy $\mu, \lambda + 2\mu > 0$. In what follows we consider the case of a homogeneous material with zero Poisson's ratio, which implies for the Lamé coefficients $\lambda = 0$ and $\mu = 1$. Then the elasticity operator with traction-free boundary conditions is defined by its energy form

$$a_\Gamma[u] := \int_{\Omega \setminus \bar{\Gamma}} \langle E(u), \Sigma(u) \rangle_{d \times d} \, dx$$

where $u \in D[a_\Gamma] := H^1(\Omega \setminus \bar{\Gamma}; \mathbb{C}^d)$. Here $\langle \cdot, \cdot \rangle_{d \times d}$ denotes the standard scalar product in $\mathbb{C}^{d \times d}$ identified with \mathbb{C}^{d^2} . Then a_Γ defines a positive quadratic form, which is closed by Korn's inequality, cf. e.g. [1] and the references therein. The corresponding self-adjoint operator A_Γ acts as the matrix differential operator

$$-\Delta - \operatorname{grad} \operatorname{div} \quad (2.3)$$

¹or more precisely the domain of its quadratic form

on functions with vanishing conormal derivative, i.e., on $\partial(\Omega^{(d)} \setminus \bar{\Gamma})$ we have

$$\Sigma(u) \cdot \mathbf{n} = 0 \quad (2.4)$$

for all $u : \Omega^{(d)} \setminus \bar{\Gamma} \rightarrow \mathbb{C}^d$ in $D(A_\Gamma)$. Here \mathbf{n} denotes the unit normal. Due to the location of the crack and the assumptions on Poisson's ratio we will use the symmetries induced by the domain and the operator. They allow us to decompose the form domain and the operator domain into symmetric pieces.

2.1. The symmetry decomposition in 2D. Using the ideas from [1, 2, 3] we denote $\mathcal{H} := L^2(\Omega^{(2)}; \mathbb{C}^2)$ and define the following subspace of symmetric waves

$$\mathcal{H}_s := \{u \in \mathcal{H} : u_1(x_1, x_2) = u_1(x_1, -x_2); u_2(x_1, x_2) = -u_2(x_1, -x_2)\}. \quad (2.5)$$

The elements of its orthogonal complement $\mathcal{H}_{as} := \mathcal{H}_s^\perp$ are called antisymmetric waves. Furthermore, we introduce

$$\mathcal{H}_1 := \left\{ u \in \mathcal{H} : \partial_2 u_1(x_1, x_2) = 0, u_2(x_1, x_2) = 0 \right\}, \quad (2.6)$$

and denote by $\mathcal{H}_2 = \mathcal{H}_s \ominus \mathcal{H}_1$ its orthogonal complement in \mathcal{H}_s . Then

$$\mathcal{H}_2 = \left\{ u \in \mathcal{H}_s : \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_1(x_1, x_2) dx_2 = 0 \text{ for a.e. } x_1 \in \mathbb{R} \right\} \quad (2.7)$$

and we obtain the following decomposition of Hilbert space

$$\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{as} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_{as}.$$

The form a_Γ as well as the operator A_Γ decompose in the same way

$$\begin{aligned} a_\Gamma &= a_\Gamma^{(s)} \oplus a_\Gamma^{(as)} = a_\Gamma^{(1)} \oplus a_\Gamma^{(2)} \oplus a_\Gamma^{(as)}, \\ A_\Gamma &= A_\Gamma^{(s)} \oplus A_\Gamma^{(as)} = A_\Gamma^{(1)} \oplus A_\Gamma^{(2)} \oplus A_\Gamma^{(as)}, \end{aligned}$$

where $a_\Gamma^{(\dagger)}$ and $A_\Gamma^{(\dagger)}$ act in \mathcal{H}_\dagger for $\dagger \in \{s, as, 1, 2\}$, cf. [1]. Based on the ideas in [15] one proves that the essential spectrum of $A_\Gamma^{(\dagger)}$ is independent of the crack. We have

$$\sigma_{\text{ess}}(A_\Gamma) = \sigma_{\text{ess}}(A_\Gamma^{(s)}) = \sigma_{\text{ess}}(A_\Gamma^{(as)}) = \sigma_{\text{ess}}(A_\Gamma^{(1)}) = [0, \infty) \quad (2.8)$$

and

$$\sigma_{\text{ess}}(A_\Gamma^{(2)}) = [\Lambda, \infty) \quad (2.9)$$

for some $\Lambda > 0$, cf. [1]. The role of Λ is elucidated in Section 4.

2.2. The symmetry decomposition in 3D. In the case of a three-dimensional plate we put $\mathcal{H} := L^2(\Omega^{(3)}; \mathbb{C}^3)$ as well as

$$\mathcal{H}_s := \{u \in \mathcal{H} : u_k(x_1, x_2, x_3) = u_k(x_1, x_2, -x_3), \ k = 1, 2 \wedge \quad (2.10)$$

$$u_3(x_1, x_2, x_3) = -u_3(x_1, x_2, -x_3)\}, \quad (2.11)$$

and $\mathcal{H}_{as} := (\mathcal{H}_s)^\perp$. We set

$$\mathcal{H}_1 := \{u \in \mathcal{H} : \partial_3 u_1 = \partial_3 u_2 = 0, \ u_3 = 0\} \quad (2.12)$$

and $\mathcal{H}_2 := \mathcal{H}_s \ominus \mathcal{H}_1$. As above we obtain the decomposition $A_\Gamma = A_\Gamma^{(1)} \oplus A_\Gamma^{(2)} \oplus A_\Gamma^{(as)}$, where $A_\Gamma^{(\dagger)}$ acts in \mathcal{H}_\dagger for $\dagger \in \{s, as, 1, 2\}$ and

$$\sigma_{\text{ess}}(A_\Gamma) = \sigma_{\text{ess}}(A_\Gamma^{(s)}) = \sigma_{\text{ess}}(A_\Gamma^{(as)}) = \sigma_{\text{ess}}(A_\Gamma^{(1)}) = [0, \infty) \quad (2.13)$$

and

$$\sigma_{\text{ess}}(A_\Gamma^{(2)}) = [\Lambda, \infty), \quad (2.14)$$

cf. [1]. The constant Λ is given as in the two-dimensional case.

3. THE RESULTS

We generalise the one-sided asymptotic estimate in [1] and prove an asymptotic formula for the eigenvalues of the operator $A_\Gamma^{(2)}$ as the cracks size tends to zero. Moreover, we show an estimate on the number of eigenvalues for small crack sizes. Since, the operator $A_\Gamma^{(2)}$ has at least two eigenvalues in the two-dimensional case and infinitely many eigenvalues in three dimensions we will need an additional symmetry conditions on the shape of the crack. In doing so, we may prove an asymptotic formula for each eigenvalue.

3.1. The two-dimensional case. We assume that $\Gamma \subseteq \mathbb{R}$ is a finite union of bounded open intervals and that Γ is symmetric with respect to the axis $x_1 = 0$, i.e., we have $\Gamma = -\Gamma$. We put $\Gamma_\ell := \ell \cdot \Gamma$.

Theorem 3.1. *There exists $\ell_0 = \ell_0(\Gamma) > 0$ such that for $\ell \in (0, \ell_0)$ the operator $A_{\Gamma_\ell}^{(2)}$ has exactly two discrete eigenvalues $\lambda_1(\ell)$ and $\lambda_2(\ell)$ below its essential spectrum $[\Lambda, \infty)$. They satisfy*

$$\Lambda - \lambda_1(\ell) = \nu_1 \cdot \ell^4 + \mathcal{O}(\ell^5) \quad \text{as } \ell \rightarrow 0, \quad (3.1)$$

$$\Lambda - \lambda_2(\ell) = \nu_2 \cdot \ell^8 + \mathcal{O}(\ell^9) \quad \text{as } \ell \rightarrow 0. \quad (3.2)$$

The constants $\nu_1 = \nu_1(\Gamma) > 0$ and $\nu_2 = \nu_2(\Gamma) > 0$ are given by (5.42) and (5.43).

3.2. The three-dimensional case. We assume that $\Gamma = B(0, 1)$ is the ball of radius 1 with centre 0 and let $\Gamma_\ell = \ell \cdot \Gamma = B(0, \ell)$ be the ball of radius ℓ and centre 0.

Theorem 3.2. *For $\ell > 0$ the elasticity operator $A_{\Gamma_\ell}^{(2)}$ has infinitely many eigenvalues below its essential spectrum $[\Lambda, \infty)$. There exists an enumeration of the eigenvalues $\lambda_m(\ell)$, $m \in \mathbb{Z}$, such that for each $m \in \mathbb{Z}$ we have the following asymptotic expansion*

$$\Lambda - \lambda_m(\ell) = \rho_m \cdot \ell^{6+4|m|} + \mathcal{O}(\ell^{7+4|m|}) \quad \text{as } \ell \rightarrow 0. \quad (3.3)$$

The constants $\rho_m > 0$ are given by (6.17).

4. THE ANALYSIS OF THE UNPERTURBED OPERATOR

As a first step we investigate the unperturbed problem corresponding to $\Gamma = \emptyset$.

4.1. The two-dimensional case. Put $\Omega = \mathbb{R} \times I$, where $I = (-\pi/2, \pi/2)$. From general regularity theory for elliptic boundary value problems we know that the domain of A_\emptyset is contained in $H^2(\Omega; \mathbb{C}^2)$, i.e., we have

$$D(A_\emptyset) = \{u \in H^2(\Omega; \mathbb{C}^2) : \partial_1 u_2 + \partial_2 u_1 = 0 \wedge \partial_2 u_2 = 0 \text{ on } \partial\Omega\}.$$

Applying the Fourier transform in the horizontal direction leads to a family of unbounded operators depending on a complex parameter $\xi \in \mathbb{C}$

$$A_\emptyset(\xi) := \begin{pmatrix} 2\xi^2 - \partial_2^2 & -i\xi\partial_2 \\ -i\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix} \quad (4.1)$$

acting in $L_2(I; \mathbb{C}^2)$ having the operator domain

$$D(A_\emptyset(\xi)) := \{u \in H^2(I; \mathbb{C}^2) : i\xi u_2 + \partial_2 u_1 = 0 \wedge \partial_2 u_2 = 0 \text{ on } \{\pm\pi/2\}\}. \quad (4.2)$$

The corresponding sesquilinear form is given by

$$\begin{aligned} a_\emptyset(\xi)[u, v] := & \int_{-\pi/2}^{\pi/2} \xi^2 (2u_1 \overline{v_1} + u_2 \overline{v_2}) + \partial_2 u_1 \cdot \overline{\partial_2 v_1} + 2\partial_2 u_2 \cdot \overline{\partial_2 v_2} \\ & - i\xi (\partial_2 u_1 \cdot \overline{v_2} - u_2 \cdot \overline{\partial_2 v_1}) \, dx_2 \end{aligned}$$

with $u, v \in D[a_\emptyset(\xi)] := H^1(I; \mathbb{C}^2)$. This definition extends to $\xi \in \mathbb{C}$ and defines a holomorphic family of type (a) in the sense of Kato, cf. [16, Chapter VII. §4, Paragraph 2]. The decomposition of \mathcal{H} into the spaces \mathcal{H}_s , \mathcal{H}_{as} , \mathcal{H}_1 and \mathcal{H}_2 induces the following decomposition in the Fourier image

$$L_2(I; \mathbb{C}^2) = h_s \oplus h_{as} = h_1 \oplus h_2 \oplus h_{as},$$

where

$$h_s := \{u \in L_2(I; \mathbb{C}^2) : u_1(x_2) = u_1(-x_2) \wedge u_2(x_2) = -u_2(-x_2)\}, \quad (4.3)$$

$$h_{as} := \{u \in L_2(I; \mathbb{C}^2) : u_1(x_2) = -u_1(-x_2) \wedge u_2(x_2) = u_2(-x_2)\}, \quad (4.4)$$

as well as

$$h_1 := \left\{ c \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} : c \in \mathbb{C} \right\}, \quad h_2 := \left\{ u \in h_s : \int_{-\pi/2}^{\pi/2} u_1(x_2) \, dx_2 = 0 \right\}. \quad (4.5)$$

The following lemma shows that the operator $A_\emptyset(\xi)$ decomposes for every $\xi \in \mathbb{C}$ into an orthogonal sum of self-adjoint operators acting in h_s and h_{as} respectively in h_1 , h_2 and h_{as} .

Lemma 4.1 (cf. [1]). *The following two assertions hold true:*

(1) For every $\xi \in \mathbb{C}$ the operator $A_{\varnothing}(\xi)$ decomposes into

$$A_{\varnothing}(\xi) = A_{\varnothing}^{(s)}(\xi) \oplus A_{\varnothing}^{(as)}(\xi) = A_{\varnothing}^{(1)}(\xi) \oplus A_{\varnothing}^{(2)}(\xi) \oplus A_{\varnothing}^{(as)}(\xi),$$

where $A_{\varnothing}^{(\dagger)}(\xi)$ acts in h_{\dagger} for $\dagger \in \{s, as, 1, 2\}$.

(2) The operator A_{\varnothing} is unitarily equivalent to the direct integral of the $A_{\varnothing}(\xi)$'s, i.e., we have

$$A_{\varnothing} \cong \int_{\mathbb{R}}^{\oplus} A_{\varnothing}(\xi) \, d\xi.$$

Moreover, for $\dagger \in \{s, as, 1, 2\}$ we have

$$A_{\varnothing}^{(\dagger)} \cong \int_{\mathbb{R}}^{\oplus} A_{\varnothing}^{(\dagger)}(\xi) \, d\xi.$$

For every $\xi \in \mathbb{C}$ the spectrum $\sigma(A_{\varnothing}(\xi))$ consists of a discrete set of eigenvalues of finite algebraic multiplicity as $D(A_{\varnothing}(\xi))$ is compactly embedded into $L_2(I; \mathbb{C}^2)$, and thus, the resolvent is compact, cf. [16, Theorem III.6.29]. Since $(a_{\varnothing}(\xi))_{\xi \in \mathbb{C}}$ forms an analytic family of type (a) the eigenvalues depend holomorphically on ξ (with the possible exception of algebraic branching points) and the direct integral representation implies

$$\sigma(A_{\varnothing}) = \bigcup_{\xi \in \mathbb{R}} \sigma(A_{\varnothing}(\xi)) \quad \text{and} \quad \sigma(A_{\varnothing}^{(\dagger)}) = \bigcup_{\xi \in \mathbb{R}} \sigma(A_{\varnothing}^{(\dagger)}(\xi)) \quad (4.6)$$

for $\dagger \in \{s, as, 1, 2\}$.

Now we consider the eigenvalue distribution of the operator $A_{\varnothing}(\xi)$ depending on the parameter $\xi \in \mathbb{R}$. The arising curves are generally referred to as the dispersion curves of A_{\varnothing} . In what follows the dispersion curve corresponding to the lowest eigenvalue of $A_{\varnothing}^{(2)}$ is of particular interest since it describes the behaviour of the unperturbed operator near the spectral threshold Λ . More generally we fix $\xi \in \mathbb{C}$ and choose $\omega \in \mathbb{C}$, $u \in H^2(I; \mathbb{C}^2)$ such that

$$A(\xi)u - \omega u = 0 \quad \text{in } I, \quad (4.7)$$

$$i\xi u_2 + \partial_2 u_1 = 0 \quad \text{on } \{\pm\pi/2\}, \quad (4.8)$$

$$\partial_2 u_2 = 0 \quad \text{on } \{\pm\pi/2\}. \quad (4.9)$$

Here and subsequently we denote by $A(\xi)$ the matrix differential operator

$$A(\xi) := \begin{pmatrix} 2\xi^2 - \partial_2^2 & -i\xi\partial_2 \\ -i\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix}$$

if no boundary conditions are specified. The assumptions on u already imply that $u \in C^\infty([-\frac{\pi}{2}, \frac{\pi}{2}]; \mathbb{C}^2)$. In order to solve (4.7)-(4.9) we have to distinguish several cases:

- (1) Let $\omega \in \mathbb{C} \setminus \{0\}$, $\omega \notin \{\xi^2, 2\xi^2\}$: A fundamental solution of the differential equation (4.7) is given by the functions

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} \beta \\ -\xi \end{pmatrix} e^{i\beta x_2}; & v^3(x_2) &:= \begin{pmatrix} \xi \\ \gamma \end{pmatrix} e^{i\gamma x_2}; \\ v^2(x_2) &:= \begin{pmatrix} -\beta \\ -\xi \end{pmatrix} e^{-i\beta x_2}; & v^4(x_2) &:= \begin{pmatrix} \xi \\ -\gamma \end{pmatrix} e^{-i\gamma x_2}, \end{aligned}$$

where

$$\beta := \sqrt{\omega - \xi^2}, \quad \gamma := \sqrt{\frac{\omega}{2} - \xi^2}.$$

Here we choose the branch of the square root function such that $z \mapsto \sqrt{z}$ is holomorphic for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\sqrt{z} > 0$ for all $z > 0$. Moreover for $z \leq 0$ we define \sqrt{z} such that $\text{Im}(\sqrt{z}) \geq 0$.

- (2) Let $\omega \in \mathbb{C} \setminus \{0\}$, $\omega = \xi^2$: In this case we have $\beta = 0$ and the fundamental solution of (4.7) reads as

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} 0 \\ 1 \end{pmatrix}; & v^3(x_2) &:= \begin{pmatrix} \xi \\ \gamma \end{pmatrix} e^{i\gamma x_2}; \\ v^2(x_2) &:= \begin{pmatrix} 1 \\ -i\xi x_2 \end{pmatrix}; & v^4(x_2) &:= \begin{pmatrix} \xi \\ -\gamma \end{pmatrix} e^{-i\gamma x_2}. \end{aligned}$$

- (3) Let $\omega \in \mathbb{C} \setminus \{0\}$, $\omega = 2\xi^2$: Then $\gamma = 0$ and a fundamental solution of (4.7) is given by

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} \beta \\ -\xi \end{pmatrix} e^{i\beta x_2}; & v^3(x_2) &:= \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\ v^2(x_2) &:= \begin{pmatrix} -\beta \\ -\xi \end{pmatrix} e^{-i\beta x_2}; & v^4(x_2) &:= \begin{pmatrix} i\xi x_2 \\ 1 \end{pmatrix}. \end{aligned}$$

- (4) Let $\omega = 0, \xi = 0$: In this case all solutions have to be linear in both components.
 (5) Let $\omega = 0, \xi \neq 0$: Then we have $\beta = \gamma = i\xi$ and we get

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} \xi \\ i\xi \end{pmatrix} e^{-\xi x_2}; & v^3(x_2) &:= \left[x_2 \begin{pmatrix} \xi \\ i\xi \end{pmatrix} + \begin{pmatrix} 0 \\ 3i \end{pmatrix} \right] e^{-\xi x_2}; \\ v^2(x_2) &:= \begin{pmatrix} \xi \\ -i\xi \end{pmatrix} e^{\xi x_2}; & v^4(x_2) &:= \left[x_2 \begin{pmatrix} \xi \\ -i\xi \end{pmatrix} + \begin{pmatrix} 0 \\ 3i \end{pmatrix} \right] e^{\xi x_2}. \end{aligned}$$

Considering the boundary conditions leads to the well-known Rayleigh-Lamb equations for the symmetric and antisymmetric part of the operator. The following lemma deals only with the symmetric part.

Lemma 4.2. *Let $\xi \in \mathbb{C}$. Then the following assertions hold true:*

- (1) *We have $\sigma(A_{\mathcal{O}}^{(1)}(\xi)) = \{2\xi^2\}$.*

- (2) *The eigenvalues of $A_{\varnothing}^{(2)}(\xi)$ are simple.*
 (3) *Let $\omega \in \mathbb{C} \setminus \{0\}$. Then $\omega \in \sigma(A_{\varnothing}^{(2)}(\xi))$ if and only if*

$$\frac{\sin\left(\beta\frac{\pi}{2}\right)}{\beta} \cos\left(\gamma\frac{\pi}{2}\right) \gamma^2 + \cos\left(\beta\frac{\pi}{2}\right) \frac{\sin\left(\gamma\frac{\pi}{2}\right)}{\gamma} \xi^2 = 0, \quad (4.10)$$

where as before

$$\beta = \sqrt{\omega - \xi^2}, \quad \gamma = \sqrt{\frac{\omega}{2} - \xi^2}.$$

Note that $0 \notin \sigma(A_{\varnothing}^{(2)}(\xi))$ for any $\xi \in \mathbb{C}$. If $\gamma \neq 0$, then a (non-normalised) eigenfunction is given by

$$x_2 \mapsto \begin{pmatrix} i\gamma^2 \beta \cos\left(\frac{\pi}{2}\gamma\right) \cos(\beta x_2) + i\xi^2 \beta \cos\left(\frac{\pi}{2}\beta\right) \cos(\gamma x_2) \\ \xi \gamma^2 \cos\left(\frac{\pi}{2}\gamma\right) \sin(\beta x_2) - \xi \beta \gamma \cos\left(\frac{\pi}{2}\beta\right) \sin(\gamma x_2) \end{pmatrix}. \quad (4.11)$$

The assertion of Lemma 4.2 is well known. In what follows for real $\xi \in \mathbb{R}$ we denote by $\zeta_1(\xi) < \zeta_2(\xi) < \dots$ the (simple) eigenvalues of the operator $A_{\varnothing}^{(2)}(\xi)$. Obviously the functions $\zeta_k(\cdot)$ are real analytic. Let

$$\Lambda := \inf\{\zeta_1(\xi) : \xi \in \mathbb{R}\}. \quad (4.12)$$

Förster and Weidl showed in [3] that $\Lambda = 1.887837 \pm 10^{-6} > 0$ and that the infimum is achieved for $\xi = \pm \varkappa$, where

$$\varkappa = 0.632138 \pm 10^{-6} > 0. \quad (4.13)$$

In particular, from (4.6) and the invariance of the essential spectrum we obtain

$$\sigma(A_{\varnothing}^{(2)}) = \sigma(A_{\Gamma}^{(2)}) = [\Lambda, \infty) \subsetneq [0, \infty) \quad (4.14)$$

for any crack $\Gamma \subseteq \mathbb{R}$, which is given by a finite union of bounded intervals.

4.2. The three-dimensional case. In this case the unperturbed elasticity operator acts as the matrix differential operator

$$-\begin{pmatrix} 2\partial_1^2 + \partial_2^2 + \partial_3^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_1^2 + 2\partial_2^2 + \partial_3^2 & \partial_2\partial_3 \\ \partial_1\partial_3 & \partial_2\partial_3 & \partial_1^2 + \partial_2^2 + 2\partial_3^2 \end{pmatrix}$$

with the operator domain

$$D(A_{\varnothing}) = \{u \in H^2(\Omega; \mathbb{C}^3) : \partial_3 u_k + \partial_k u_3 = 0 \text{ on } \partial\Omega \text{ for } k = 1, 2, 3\}.$$

Applying the Fourier transform with respect to the first two variables we obtain a family of sectorial operators $(A_{\varnothing}(\xi))_{\xi \in \mathbb{R}^2}$ acting on $L^2(I; \mathbb{C}^3)$,

$$A_{\varnothing}(\xi) := \begin{pmatrix} 2\xi_1^2 + \xi_2^2 - \partial_3^2 & \xi_1\xi_2 & -i\xi_1\partial_3 \\ \xi_1\xi_2 & \xi_1^2 + 2\xi_2^2 - \partial_3^2 & -i\xi_2\partial_3 \\ -i\xi_1\partial_3 & -i\xi_2\partial_3 & \xi_1^2 + \xi_2^2 - 2\partial_3^2 \end{pmatrix}. \quad (4.15)$$

The domain of the operator $D(A_{\varnothing}(\xi))$ consists of those functions $u \in H^2(I; \mathbb{C}^3)$, which satisfy

$$\begin{pmatrix} \partial_3 u_1 + i\xi_1 u_3 \\ \partial_3 u_2 + i\xi_2 u_3 \\ \partial_3 u_3 \end{pmatrix} (x_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } x_3 = \pm \frac{\pi}{2}. \quad (4.16)$$

As above we define

$$h_s := \{u \in L_2(I; \mathbb{C}^3) : u_k(x_3) = u_k(-x_3) \text{ for } k = 1, 2 \wedge u_3(x_3) = -u_3(-x_3)\}, \quad (4.17)$$

$$h_{as} := \{u \in L_2(I; \mathbb{C}^3) : u_k(x_3) = -u_k(-x_3) \text{ for } k = 1, 2 \wedge u_3(x_3) = u_3(-x_3)\}, \quad (4.18)$$

as well as

$$h_1 := \left\{ c \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : c \in \mathbb{C} \right\}, \quad h_2 := \left\{ u \in h_s : \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_k(x_3) \, dx_3 = 0, \, k = 1, 2 \right\}. \quad (4.19)$$

The operator $A_{\varnothing}(\xi)$ admits the following decomposition

$$A_{\varnothing}(\xi) = A_{\varnothing}^{(s)}(\xi) \oplus A_{\varnothing}^{(as)}(\xi) = A_{\varnothing}^{(1)}(\xi) \oplus A_{\varnothing}^{(2)}(\xi) \oplus A_{\varnothing}^{(as)}(\xi),$$

where $A_{\varnothing}^{(\dagger)}(\xi)$ acts in h_{\dagger} for $\dagger \in \{s, as, 1, 2\}$. We have

$$A_{\varnothing} \cong \int_{\mathbb{R}}^{\oplus} A_{\varnothing}(\xi) \, d\xi \quad \text{and} \quad A_{\varnothing}^{(\dagger)} \cong \int_{\mathbb{R}}^{\oplus} A_{\varnothing}^{(\dagger)}(\xi) \, d\xi$$

for $\dagger \in \{s, as, 1, 2\}$.

Lemma 4.3. *For $\xi \in \mathbb{R}^2$ and $M \in \text{SO}(2)$ the following assertions hold true:*

(1) *Let $u = (u_1, u_2, u_3) \in H^2(I; \mathbb{C}^3)$. Then*

$$A_{\varnothing}(M\xi) \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} u = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} A_{\varnothing}(\xi) u. \quad (4.20)$$

(2) *We have $\sigma(A_{\varnothing}^{(2)}(M\xi)) = \sigma(A_{\varnothing}^{(2)}(\xi))$. More precisely, ω is an eigenvalue of the operator $A_{\varnothing}^{(2)}(\xi)$ if and only if*

$$\frac{\sin(\beta \frac{\pi}{2})}{\beta} \cos\left(\gamma \frac{\pi}{2}\right) \gamma^2 + \cos\left(\beta \frac{\pi}{2}\right) \frac{\sin(\gamma \frac{\pi}{2})}{\gamma} |\xi|^2 = 0, \quad (4.21)$$

where

$$\beta = \sqrt{\omega - |\xi|^2}, \quad \gamma = \sqrt{\frac{\omega}{2} - |\xi|^2}.$$

Lemma 4.3 implies that the eigenvalues of the operator $A_{\varnothing}^{(2)}(\xi)$ depend only on the norm of $\xi \in \mathbb{R}^2$ and coincide with the eigenvalues of the operator $A_{\varnothing}^{(2)}(|\xi|)$ arising in two

dimensions. For $\xi = (|\xi| \cos \alpha, |\xi| \sin \alpha)^T$, $\alpha \in [0, 2\pi)$, and $\gamma \neq 0$ the eigenfunctions of the operator $A_{\varnothing}^{(2)}(\xi)$ are given by

$$x_3 \mapsto \begin{pmatrix} M_\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i\gamma^2 \beta \cos(\frac{\pi}{2}\gamma) \cos(\beta x_3) + i|\xi|^2 \beta \cos(\frac{\pi}{2}\beta) \cos(\gamma x_3) \\ 0 \\ |\xi| \gamma^2 \cos(\frac{\pi}{2}\gamma) \sin(\beta x_3) - |\xi| \beta \gamma \cos(\frac{\pi}{2}\beta) \sin(\gamma x_3) \end{pmatrix}, \quad (4.22)$$

where we put

$$M_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \in \text{SO}(2).$$

Recall that

$$x_2 \mapsto \begin{pmatrix} i\gamma^2 \beta \cos(\frac{\pi}{2}\gamma) \cos(\beta x_2) + i|\xi|^2 \beta \cos(\frac{\pi}{2}\beta) \cos(\gamma x_2) \\ |\xi| \gamma^2 \cos(\frac{\pi}{2}\gamma) \sin(\beta x_2) - |\xi| \beta \gamma \cos(\frac{\pi}{2}\beta) \sin(\gamma x_2) \end{pmatrix}$$

is an eigenfunction of the operator $A_{\varnothing}^{(2)}(|\xi|)$. In particular, we obtain for the three-dimensional plate

$$\sigma_{\text{ess}}(A_{\varnothing}^{(2)}) = [\Lambda, \infty),$$

with the same constant $\Lambda > 0$ as for the infinite strip.

5. PROOF OF THE MAIN RESULT - 2D

5.1. An equivalent mixed problem. The aim of this section is the construction of the Dirichlet-to-Neumann operator corresponding to our problem. For this purpose we provide the boundary data on the crack and construct the solution of the inhomogeneous problem on the upper part of the strip $\Omega_+ := \mathbb{R} \times I_+$ with $I_+ := (0, \frac{\pi}{2})$. This will be sufficient as we are only interested in the symmetric part of the operator. The latter is unitarily equivalent to the operator $A_{\Gamma+}$, which is induced by the quadratic form

$$a_{\Gamma+}[u] := \int_{\Omega_+} \langle E(u), \Sigma(u) \rangle_{2 \times 2} \, dx \quad (5.1)$$

with $u \in D[a_{\Gamma+}] := \{u \in H^1(\Omega_+; \mathbb{C}^2) : \text{supp}(u_2|_{\mathbb{R} \times \{0\}}) \subseteq \overline{\Gamma}\}$. Here $E(u)$ and $\Sigma(u)$ are given as in (2.1) and (2.2). Then $A_{\Gamma+}$ acts as the elasticity operator $-\Delta - \text{grad div}$ and its domain contains exactly those functions $u \in H^1(\Omega_+; \mathbb{C}^2)$ which satisfy

$$\begin{cases} \partial_1 u_2 + \partial_2 u_1 = 0 & \text{on } \mathbb{R} \times \{\frac{\pi}{2}\}, \\ 2\partial_2 u_2 = 0 & \text{on } \mathbb{R} \times \{\frac{\pi}{2}\}, \end{cases} \quad (5.2)$$

and

$$\begin{cases} -(\partial_1 u_2 + \partial_2 u_1) = 0 & \text{on } \mathbb{R} \times \{0\}, \\ -2\partial_2 u_2 = 0 & \text{on } \Gamma, \\ u_2 = 0 & \text{on } (\mathbb{R} \times \{0\}) \setminus \overline{\Gamma}. \end{cases} \quad (5.3)$$

As before these identities should be understood in the weak sense. As a particular consequence we have reduced the original problem to a mixed problem. The Hilbert space $\mathcal{H}_+ := L_2(\Omega_+; \mathbb{C}^2)$ decomposes into an orthogonal sum

$$\mathcal{H}_+ = \mathcal{H}_{1+} \oplus \mathcal{H}_{2+}$$

with

$$\mathcal{H}_{2+} := \left\{ u \in L_2(\Omega_+; \mathbb{C}^2) : \int_0^{\frac{\pi}{2}} u_1(x_1, x_2) dx_2 = 0 \text{ for a.e. } x_1 \in \mathbb{R} \right\} \quad (5.4)$$

and $\mathcal{H}_{1+} := (\mathcal{H}_{2+})^\perp$. The form and the operator decompose in the same way in an orthogonal sum

$$a_{\Gamma+} = a_{\Gamma+}^{(1)} \oplus a_{\Gamma+}^{(2)}, \quad A_{\Gamma+} = A_{\Gamma+}^{(1)} \oplus A_{\Gamma+}^{(2)},$$

where $a_{\Gamma+}^{(\dagger)}$ and $A_{\Gamma+}^{(\dagger)}$ acts in $\mathcal{H}_{\dagger+}$ for $\dagger \in \{1, 2\}$. Next we define the analogues of the spaces h_i , $i = 1, 2$ which were introduced in (4.5). We denote by h_{1+} the linear span of the constant function $(1, 0)^T$ and let $h_{2+} = h_{1+}^\perp$,

$$h_{2+} = \left\{ u \in L_2(I_+; \mathbb{C}^2) : \int_0^{\frac{\pi}{2}} u_1(x_2) dx_2 = 0 \right\}. \quad (5.5)$$

The Fourier transform in the horizontal direction leads as before to the parameter-dependent operator

$$A_{\varnothing+}(\xi) := \begin{pmatrix} 2\xi^2 - \partial_2^2 & -i\xi\partial_2 \\ -i\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix}$$

which acts in the Hilbert space $L_2(I_+; \mathbb{C}^2)$ having the operator domain

$$D(A_{\varnothing+}(\xi)) := \{u \in H^2(I_+; \mathbb{C}^2) : i\xi u_2 + \partial_2 u_1 = 0 \wedge \partial_2 u_2 = 0 \text{ on } \{0, \pi/2\}\}.$$

We note that the subspaces h_{1+} and h_{2+} reduce the operator $A_{\varnothing+}(\xi)$, giving rise to self-adjoint operators $A_{\varnothing+}^{(1)}(\xi)$ and $A_{\varnothing+}^{(2)}(\xi)$. Clearly, the operators $A_{\varnothing+}^{(i)}(\xi)$ and $A_{\varnothing}^{(i)}(\xi)$, $i = 1, 2$, are unitarily equivalent.

5.2. The construction of the Dirichlet-to-Neumann operator. In the first instance we consider $\omega \in \mathbb{C} \setminus \{0\}$ and let $g \in H^{1/2}(\mathbb{R})$. We want to search for a solution $u \in H^1(\Omega_+; \mathbb{C}^2)$ of the eigenvalue problem

$$(-\Delta - \text{grad div})u = \omega u, \quad \text{in } \Omega_+, \quad (5.6)$$

which satisfies the boundary data

$$\begin{cases} \partial_2 u_1 + \partial_1 u_2 = 0 & \text{on } \mathbb{R} \times \{\frac{\pi}{2}\}, \\ 2\partial_2 u_2 = 0 & \text{on } \mathbb{R} \times \{\frac{\pi}{2}\}, \end{cases} \quad (5.7)$$

$$\begin{cases} \partial_2 u_1 + \partial_1 u_2 = 0 & \text{on } \mathbb{R} \times \{0\}, \\ u_2 = g & \text{on } \mathbb{R} \times \{0\}, \end{cases} \quad (5.8)$$

Note that conditions (5.7)-(5.8) should be understood in their variational form, i.e., the assertions (5.6)-(5.8) are equivalent to

$$\int_{\Omega_+} \langle E(u), \Sigma(v) \rangle_{2 \times 2} dx = \omega \langle u, v \rangle_{\Omega_+}, \quad (5.9)$$

which should hold for all $v \in H^1(\Omega_+; \mathbb{C}^2)$ with $v_2|_{\mathbb{R} \times \{0\}} = 0$. Let \hat{u} be the Fourier transform of u taken in the horizontal direction. Then we have for $(\xi, x_2) \in \mathbb{R} \times I_+$

$$A(\xi)\hat{u}(\xi, x_2) = \omega\hat{u}(\xi, x_2), \quad (5.10)$$

as well as

$$\begin{cases} \partial_2 \hat{u}_1(\xi, \frac{\pi}{2}) + i\xi \hat{u}_2(\xi, \frac{\pi}{2}) = 0, & \xi \in \mathbb{R} \\ 2\partial_2 \hat{u}_2(\xi, \frac{\pi}{2}) = 0, & \xi \in \mathbb{R} \end{cases} \quad (5.11)$$

$$\begin{cases} \partial_2 \hat{u}_1(\xi, 0) + i\xi \hat{u}_2(\xi, 0) = 0, & \xi \in \mathbb{R} \\ \hat{u}_2(\xi, 0) = \hat{g}(\xi), & \xi \in \mathbb{R}. \end{cases} \quad (5.12)$$

Note that this already implies $\hat{u}(\xi, \cdot) \in C^\infty([0, \frac{\pi}{2}]; \mathbb{C}^2)$ for almost every $\xi \in \mathbb{R}$. Since $\omega \neq 0$ the solution \hat{u} is for $\xi^2 \notin \{\omega, \frac{\omega}{2}\}$ a linear combination of

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} \beta \\ -\xi \end{pmatrix} e^{i\beta x_2}; & v^3(x_2) &:= \begin{pmatrix} \xi \\ \gamma \end{pmatrix} e^{i\gamma x_2}; \\ v^2(x_2) &:= \begin{pmatrix} -\beta \\ -\xi \end{pmatrix} e^{-i\beta x_2}; & v^4(x_2) &:= \begin{pmatrix} \xi \\ -\gamma \end{pmatrix} e^{-i\gamma x_2}, \end{aligned}$$

where

$$\beta = \sqrt{\omega - \xi^2} \quad \text{and} \quad \gamma = \sqrt{\frac{\omega}{2} - \xi^2}.$$

Thus, we obtain for $(\xi, x_2) \in \mathbb{R} \times I_+$

$$\hat{u}(\xi, x_2) = \sum_{k=1}^4 a_k(\xi, \omega) v^k(x_2)$$

with coefficients $a_1(\xi, \omega), \dots, a_4(\xi, \omega)$. Inserting the boundary conditions leads to a linear system $L(\xi, \omega)a(\xi, \omega) = b(\xi)$, where

$$L(\xi, \omega) := \begin{pmatrix} (\beta^2 - \xi^2) e^{i\beta \frac{\pi}{2}} & (\beta^2 - \xi^2) e^{-i\beta \frac{\pi}{2}} & 2\gamma\xi e^{i\gamma \frac{\pi}{2}} & -2\gamma\xi e^{-i\gamma \frac{\pi}{2}} \\ -2\beta\xi e^{i\beta \frac{\pi}{2}} & 2\beta\xi e^{-i\beta \frac{\pi}{2}} & 2\gamma^2 e^{i\gamma \frac{\pi}{2}} & 2\gamma^2 e^{-i\gamma \frac{\pi}{2}} \\ i(\beta^2 - \xi^2) & i(\beta^2 - \xi^2) & 2i\gamma\xi & -2i\gamma\xi \\ -\xi & -\xi & \gamma & -\gamma \end{pmatrix} \quad (5.13)$$

and

$$a(\xi, \omega) := \begin{pmatrix} a_1(\xi, \omega) \\ a_2(\xi, \omega) \\ a_3(\xi, \omega) \\ a_4(\xi, \omega) \end{pmatrix}, \quad b(\xi) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hat{g}(\xi) \end{pmatrix}.$$

Thus, $a(\xi, \omega) = L(\xi, \omega)^{-1}b(\xi)$ provided $L(\xi)$ is invertible. We note that the determinant of $L(\xi)$ is given by

$$\det(L(\xi, \omega)) = 32\gamma^2(\gamma^2 + \xi^2) \left[\sin\left(\beta\frac{\pi}{2}\right) \cos\left(\gamma\frac{\pi}{2}\right) \gamma^3 + \cos\left(\beta\frac{\pi}{2}\right) \sin\left(\gamma\frac{\pi}{2}\right) \beta\xi^2 \right],$$

which coincides up to a factor with the left-hand side of the Rayleigh-Lamb equation (4.10). In particular if $\omega \in \mathbb{C} \setminus [0, \infty)$ then $L(\xi, \omega)$ is invertible for every $\xi \in \mathbb{R}$. Put

$$h(x) = -2\partial_2 u_2(x, 0), \quad x \in \mathbb{R}.$$

Then its Fourier transform \hat{h} satisfies $\hat{h}(\xi) = R(\xi, \omega)a(\xi, \omega)$, where

$$R(\xi, \omega) = \begin{pmatrix} 2i\beta\xi & -2i\beta\xi & -2i\gamma^2 & -2i\gamma^2 \end{pmatrix}.$$

An elementary calculation shows that

$$\hat{h}(\xi) = m_\omega(\xi)\hat{g}(\xi)$$

where

$$m_\omega(\xi) := \frac{-2 \sin\left(\frac{\beta\pi}{2}\right) \sin\left(\frac{\gamma\pi}{2}\right) [\gamma^6 + 2\gamma^2\xi^4 + \xi^6] + 4 \left[\cos\left(\frac{\beta\pi}{2}\right) \cos\left(\frac{\gamma\pi}{2}\right) - 1 \right] \beta\gamma^3\xi^2}{(\gamma^2 + \xi^2) \left[\sin\left(\frac{\beta\pi}{2}\right) \cos\left(\frac{\gamma\pi}{2}\right) \gamma^3 + \cos\left(\frac{\beta\pi}{2}\right) \sin\left(\frac{\gamma\pi}{2}\right) \beta\xi^2 \right]}, \quad (5.14)$$

As before β and γ depend on ξ and we have $\beta = \sqrt{\omega - \xi^2}$ and $\gamma = \sqrt{\frac{\omega}{2} - \xi^2}$. As we do not need this explicit representation of m_ω for $\omega \neq 0$, we do not want to give the separate steps of the calculation. Now we consider the case $\omega = 0$. A fundamental solution of the differential equation

$$A(\xi)\hat{u}(\xi, \cdot) = 0$$

is given by the functions

$$\begin{aligned} v^1(x_2) &:= \begin{pmatrix} \xi \\ i\xi \end{pmatrix} e^{-\xi x_2}; & v^3(x_2) &:= \left[x_2 \begin{pmatrix} \xi \\ i\xi \end{pmatrix} + \begin{pmatrix} 0 \\ 3i \end{pmatrix} \right] e^{-\xi x_2}; \\ v^2(x_2) &:= \begin{pmatrix} \xi \\ -i\xi \end{pmatrix} e^{\xi x_2}; & v^4(x_2) &:= \left[x_2 \begin{pmatrix} \xi \\ -i\xi \end{pmatrix} + \begin{pmatrix} 0 \\ 3i \end{pmatrix} \right] e^{\xi x_2}. \end{aligned}$$

For the matrix $L(\xi, 0)$ we obtain

$$L(\xi, 0) = \begin{pmatrix} -2\xi^2 e^{-\xi\frac{\pi}{2}} & 2\xi^2 e^{\xi\frac{\pi}{2}} & (-2\xi - \pi\xi^2) e^{-\xi\frac{\pi}{2}} & (-2\xi + \pi\xi^2) e^{\xi\frac{\pi}{2}} \\ -2i\xi^2 e^{-\xi\frac{\pi}{2}} & -2i\xi^2 e^{\xi\frac{\pi}{2}} & -i(4\xi + \pi\xi^2) e^{-\xi\frac{\pi}{2}} & i(4\xi - \pi\xi^2) e^{\xi\frac{\pi}{2}} \\ -2\xi^2 & 2\xi^2 & -2\xi & -2\xi \\ i\xi & -i\xi & 3i & 3i \end{pmatrix}. \quad (5.15)$$

Calculating explicitly its inverse we obtain for $\xi \in \mathbb{R}$

$$L(\xi, 0)^{-1} = \begin{pmatrix} \dots & \dots & \dots & \frac{ie^{\pi\xi}(-2+2e^{\pi\xi}+2\pi\xi+\pi^2\xi^2)}{4\xi(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ \dots & \dots & \dots & \frac{i(2+2e^{\pi\xi}(-2-2\pi\xi+\pi^2\xi^2))}{4\xi(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ \dots & \dots & \dots & -\frac{ie^{\pi\xi}(-1+e^{\pi\xi}+\pi\xi)}{2(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ \dots & \dots & \dots & -\frac{i(-1+e^{\pi\xi}(1+\pi\xi))}{2(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \end{pmatrix}.$$

For the function \hat{u} we have

$$\hat{u}(\xi, x_2) = \hat{g}(\xi) \cdot \begin{pmatrix} v^1(x_2) \\ v^2(x_2) \\ v^3(x_2) \\ v^4(x_2) \end{pmatrix}^T \cdot \begin{pmatrix} \frac{ie^{\pi\xi}(-2+2e^{\pi\xi}+2\pi\xi+\pi^2\xi^2)}{4\xi(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ \frac{i(2+2e^{\pi\xi}(-2-2\pi\xi+\pi^2\xi^2))}{4\xi(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ -\frac{ie^{\pi\xi}(-1+e^{\pi\xi}+\pi\xi)}{2(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \\ -\frac{i(-1+e^{\pi\xi}(1+\pi\xi))}{2(-1+e^{2\pi\xi}+2e^{\pi\xi}+2\pi\xi e^{\pi\xi})} \end{pmatrix}.$$

In particular, we observe that $\hat{u}(\xi)$ does not have any singularities for $\xi \in \mathbb{R}$. Since $R(\xi, 0) = (2i\xi^2 \quad 2i\xi^2 \quad 4i\xi \quad -4i\xi)$, we have

$$-2\partial_2 \hat{u}_2(\xi, 0) = m_0(\xi) \hat{g}(\xi),$$

where

$$m_0(\xi) := \xi \cdot \frac{\cosh(\pi\xi) - \left(1 + \frac{\pi^2\xi^2}{2}\right)}{\sinh(\pi\xi) + \pi\xi}. \quad (5.16)$$

Until now we did not use the symmetry condition corresponding to the space \mathcal{H}_2 . Imposing this additional symmetry we observe that $\hat{u}(\xi, \cdot) \in h_{2+}$ for almost every $\xi \in \mathbb{R}$, and thus,

$$\begin{pmatrix} e^{i\beta\frac{\pi}{2}} - 1 \\ e^{-i\beta\frac{\pi}{2}} - 1 \\ \frac{\xi}{\gamma} \left(e^{i\gamma\frac{\pi}{2}} - 1 \right) \\ -\frac{\xi}{\gamma} \left(e^{-i\gamma\frac{\pi}{2}} - 1 \right) \end{pmatrix}^T a(\xi, \omega) = 0$$

if $\omega \neq 0$. An easy calculation shows that this condition is always satisfied. This is due to the fact that the only solution of (5.6)-(5.8) in \mathcal{H}_{1+} is the trivial one. Moreover, if $\omega \in [0, \Lambda)$ the boundary value problem (5.6)-(5.8) is not Fredholm. However, in the following lemma we observe that it becomes Fredholm by imposing the additional symmetry condition.

Lemma 5.1. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$. Then for every $g \in H^{1/2}(\mathbb{R})$ there exists a unique $u \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$ which solves (5.6)-(5.8), moreover we have $\|u\|_{H^1(\Omega_+; \mathbb{C}^2)} \leq C(\omega)\|g\|_{H^{1/2}(\mathbb{R})}$ for some $C(\omega) > 0$ independent of g .*

Proof. Applying the Fourier transform leads to the operator pencil

$$\mathfrak{A}(\xi) = \begin{pmatrix} A(\xi) - \omega \\ B_1(\xi) \\ \vdots \\ B_4(\xi) \end{pmatrix} : H^2(I_+; \mathbb{C}^2) \rightarrow \begin{matrix} L_2(I_+; \mathbb{C}^2) \\ \oplus \\ \mathbb{C}^4 \end{matrix},$$

where

$$A(\xi) = \begin{pmatrix} 2\xi^2 - \partial_2^2 & -i\xi\partial_2 \\ -i\xi\partial_2 & \xi^2 - 2\partial_2^2 \end{pmatrix}$$

and

$$\begin{aligned} B_1(\xi)u &= \partial_2 u_1 \left(\frac{\pi}{2} \right) + i\xi u_2 \left(\frac{\pi}{2} \right), & B_2(\xi)u &= 2\partial_2 u_2 \left(\frac{\pi}{2} \right), \\ B_3(\xi)u &= \partial_2 u_1(0) + i\xi u_2(0), & B_4(\xi)u &= u_2(0). \end{aligned}$$

It is well known that for every $\xi \in \mathbb{R}$ the operator $\mathfrak{A}(\xi)$ is a Fredholm operator with Fredholm index 0 as it corresponds to a self-adjoint problem on a bounded domain, cf. [17, Chapter 4] or [18, Chapter 3]. We consider its restriction

$$\mathfrak{A}(\xi)|_{h_{2+}} : H^2(I_+; \mathbb{C}^2) \cap h_{2+} \rightarrow h_{2+} \oplus \mathbb{C}^4.$$

A short calculation shows that $\mathfrak{A}(\xi)|_{h_{2+}}$ is also Fredholm with Fredholm index 0. For $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ we have $\ker(\mathfrak{A}(\xi)|_{h_{2+}}) = \{0\}$ for all $\xi \in \mathbb{R}$, and thus, $\mathfrak{A}(\xi)|_{h_{2+}}$ is invertible for all $\xi \in \mathbb{R}$. Adapting slightly the proof of [18, Theorem 5.3.2] the invertibility of the restricted pencil shows that the Poisson problem is uniquely solvable for $g \in H^{1/2}(\mathbb{R})$. \square

Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$. We denote by $K_\omega : H^{1/2}(\mathbb{R}) \rightarrow H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$ the Poisson operator which maps the boundary value g on the solution u of the Poisson problem (5.6)-(5.8).

Lemma 5.2. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ and $s \in \mathbb{R}$. Then the operator K_ω extends to a bounded linear mapping $K_\omega : H^s(\mathbb{R}) \rightarrow H^{s+1/2}(\Omega_+; \mathbb{C}^2)$.*

The proof follows easily by using a variant of [18, Theorem 5.3.2] and restricting ourselves to functions u which are contained in \mathcal{H}_{2+} . The Dirichlet-to-Neumann operator $D_\omega : H^{1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\mathbb{R})$ is defined by

$$\widehat{D_\omega g}(\xi) := \hat{g}(\xi) \cdot m_\omega(\xi).$$

It satisfies

$$D_\omega : H^s(\mathbb{R}) \rightarrow H^{s-1}(\mathbb{R}) \quad \text{for any } s \in \mathbb{R},$$

cf. also [18, Theorem 5.3.2]. The following lemmas give a variational characterisation of the Poisson and the Dirichlet-to-Neumann operators. Their proof is based on a simple integration by parts argument after applying the Fourier transform in the horizontal direction.

Lemma 5.3. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ and $g \in H^{1/2}(\mathbb{R})$. For $u \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$ the following conditions are equivalent:*

- (1) $K_\omega g = u$.
- (2) *The function u satisfies $u_2|_{\mathbb{R} \times \{0\}} = g$ and for all $v \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$ with $v_2|_{\mathbb{R} \times \{0\}} = 0$ we have*

$$\int_{\Omega_+} \langle E(u), \Sigma(v) \rangle_{2 \times 2} dx = \omega \langle u, v \rangle_{\Omega_+}. \quad (5.17)$$

Lemma 5.4. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ and $g \in H^{1/2}(\mathbb{R})$, $u = K_\omega g \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$. Then for $h \in H^{-1/2}(\mathbb{R})$ the following two assertions are equivalent:*

- (1) $D_\omega g = h$.
- (2) *For all $v \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$ we have*

$$\int_{\Omega_+} \langle E(u), \Sigma(v) \rangle_{2 \times 2} dx = \omega \langle u, v \rangle_{\Omega} + \langle h, v_2(\cdot, 0) \rangle_{\mathbb{R}}. \quad (5.18)$$

Next we prove a perturbation formula for the Dirichlet-to-Neumann operator with respect to the spectral parameter ω , which will be essential for the proof of Theorem 3.1.

Theorem 5.5. *Let $\omega, \eta \in \mathbb{C} \setminus [\Lambda, \infty)$. Then the following identities hold true:*

- (1) $K_\omega = (I + (\omega - \eta)(A_{\emptyset+}^{(2)} - \omega)^{-1})K_\eta$;
- (2) $D_\omega^* = D_{\bar{\omega}}$;
- (3) $D_\omega = D_{\bar{\eta}} - (\omega - \bar{\eta})K_\eta^*K_\omega$;
- (4) $D_\omega = D_{\bar{\eta}} - (\omega - \bar{\eta})K_\eta^* \left(I + (\omega - \eta)(A_{\emptyset+}^{(2)} - \omega)^{-1} \right) K_\eta$.

We note that $K_\omega \in \mathcal{L}(H^{-1/2}(\mathbb{R}), L_2(\Omega_+; \mathbb{C}^2))$, and thus, its adjoint satisfies $K_\omega^* \in \mathcal{L}(L_2(\Omega_+; \mathbb{C}^2), H^{1/2}(\mathbb{R}))$.

A more general assertion is given in [19, Proposition 2.6] in the context of boundary triplets.

Proof. The proof is based on variational arguments. We note that (4) follows by combining assertions (1) and (3). Let $g \in H^{1/2}(\mathbb{R})$ and

$$u := K_\eta g + (\omega - \eta)(A_{\emptyset+}^{(2)} - \omega)^{-1}K_\eta g.$$

Then $u(x, 0) = g$ and for $v \in H^1(\Omega_+; \mathbb{C}^2) \cap \mathcal{H}_{2+}$, $v_2|_{\mathbb{R} \times \{0\}} = 0$ we have

$$\begin{aligned} \int_{\Omega_+} \langle E(u), \Sigma(v) \rangle_{2 \times 2} dx &= \eta \langle K_\eta g, v \rangle_{\Omega_+} + (\omega - \eta) \cdot a_{0+} [(A_{\emptyset+}^{(2)} - \omega)^{-1} K_\eta g, v] \\ &= \eta \langle K_\eta g, v \rangle_{\Omega_+} + (\omega - \eta) \cdot \langle A_{\emptyset+}^{(2)} (A_{\emptyset+}^{(2)} - \omega)^{-1} K_\eta g, v \rangle_{\Omega_+} \\ &= \omega \langle K_\eta g, v \rangle_{\Omega_+} + \omega(\omega - \eta) \langle (A_{\emptyset+}^{(2)} - \omega)^{-1} K_\eta g, v \rangle_{\Omega_+} \\ &= \omega \langle u, v \rangle_{\Omega_+}. \end{aligned}$$

Now Lemma 5.4 implies the first assertion. Next we prove assertions (2) and (3). For $g, h \in H^{1/2}(\mathbb{R})$ and $u := K_\omega g$, $v := K_\eta h$ we have

$$\langle D_\omega g, h \rangle_{\mathbb{R}} - \langle g, D_\eta h \rangle_{\mathbb{R}} = (\omega - \bar{\eta}) \langle u, v \rangle_{\Omega_+} = (\omega - \bar{\eta}) \langle g, K_\omega^* K_\eta v \rangle_{\Omega_+}.$$

This proves (2) and (3). \square

Now we return to the mixed problem and introduce the truncated operator acting only on functions, which are supported on $\bar{\Gamma}$. We define

$$\tilde{H}_0^{1/2}(\Gamma) := \{g \in H^{1/2}(\mathbb{R}) : \text{supp}(g) \subseteq \bar{\Gamma}\}, \quad (5.19)$$

$$H^{-1/2}(\Gamma) := \{g \in (C_c^\infty(\Gamma))' : \exists G \in H^{-1/2}(\mathbb{R}) \text{ such that } g = G|_\Gamma\}. \quad (5.20)$$

Here $C_c^\infty(\Gamma)$ is the space of smooth functions with compact support contained in Γ ; its dual $(C_c^\infty(\Gamma))'$ is the space of distributions on Γ . We note that $\tilde{H}_0^{1/2}(\Gamma)$ is a closed subspace of distributions on \mathbb{R} whereas $H^{-1/2}(\Gamma)$ is a subspace of distributions on Γ . The latter space may be identified with the quotient space

$$H^{-1/2}(\mathbb{R}) / \tilde{H}_0^{-1/2}(\mathbb{R} \setminus \bar{\Gamma}),$$

where $\tilde{H}_0^{-1/2}(\mathbb{R} \setminus \bar{\Gamma})$ contains, by definition, those distributions in $H^{-1/2}(\mathbb{R})$ which have support in $\mathbb{R} \setminus \bar{\Gamma}$. We endow the spaces in (5.19) and (5.20) with their natural topology, i.e., $\tilde{H}_0^{1/2}(\Gamma)$ carries the subspace topology of $H^{1/2}(\mathbb{R})$ and $H^{-1/2}(\Gamma)$ has the quotient topology. Note that we may identify $\tilde{H}_0^{1/2}(\Gamma)$ with the subspace of $L_2(\Gamma)$ which consists of those functions whose extension by 0 yields an element of $H^{1/2}(\mathbb{R})$. Furthermore, the space $\tilde{H}_0^{1/2}(\Gamma)$ is an isometric realisation of the (anti-)dual of $H^{-1/2}(\Gamma)$ and vice-versa. The dual pairing is given by the expression

$$\langle g, h \rangle_\Gamma := \langle G, h \rangle_{\mathbb{R}}, \quad g \in H^{-1/2}(\Gamma), \quad h \in \tilde{H}_0^{1/2}(\Gamma), \quad (5.21)$$

where $G \in H^{-1/2}(\mathbb{R})$ denotes any extension of g , cf. [17, Theorem 3.14]. In particular (5.21) is independent of the chosen extensions G which is due to the fact, that $C_c^\infty(\Gamma)$ is a dense subset of $\tilde{H}_0^{1/2}(\Gamma)$, cf. [17, Theorem 3.29]. Thus, the domain of the quadratic form $a_{\Gamma+}$ may be rewritten as follows

$$D[a_{\Gamma+}] = \left\{ u \in H^1(\Omega_+; \mathbb{C}^2) : u_2|_{\mathbb{R} \times \{0\}} \in \tilde{H}_0^{1/2}(\Gamma) \right\}. \quad (5.22)$$

Then we define the truncated Dirichlet-to-Neumann operator

$$D_{\Gamma, \omega} : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma), \quad D_{\Gamma, \omega} := r_\Gamma D_\omega e_\Gamma, \quad (5.23)$$

where $r_\Gamma : H^{-1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\Gamma)$ is the restriction operator and $e_\Gamma : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{1/2}(\mathbb{R})$ is the embedding. Identifying $\tilde{H}_0^{1/2}(\Gamma)$ with a subspace of $L_2(\Gamma)$, the operator e_Γ is simply extension by 0.

Theorem 5.6. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$. Then*

$$\dim \ker(A_\Gamma^{(2)} - \omega) = \dim \ker(D_{\Gamma, \omega}). \quad (5.24)$$

The proof is an easy consequence of the variational characterisation of the Dirichlet-to-Neumann operator in Lemma 5.4 and the duality of $\tilde{H}_0^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. We note that

$$D_{\Gamma,\omega} : \tilde{H}_0^{1/2}(\Gamma) \rightarrow \tilde{H}_0^{1/2}(\Gamma)^*,$$

and thus, the operator $D_{\Gamma,\omega}$ may be completely described by its sesquilinear form

$$d_{\Gamma,\omega}[g, h] := \langle D_{\Gamma,\omega}g, h \rangle_{H^{-1/2}(\Gamma), \tilde{H}_0^{1/2}(\Gamma)} \quad (5.25)$$

where $g, h \in D[d_{\Gamma,\omega}] := \tilde{H}_0^{1/2}(\Gamma)$. Using the dual pairing (5.21) we have for $g, h \in \tilde{H}_0^{1/2}(\Gamma)$

$$d_{\Gamma,\omega}[g, h] = \langle D_\omega g, h \rangle_{\mathbb{R}} = \int_{\mathbb{R}} m_\omega(\xi) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} d\xi \quad (5.26)$$

since $D_\omega g \in H^{-1/2}(\mathbb{R})$ is an extension of $D_{\Gamma,\omega}g \in H^{-1/2}(\Gamma)$. Lemma 5.4 implies that

$$d_{\Gamma,\omega}[g, h] = \langle D_\omega g, h \rangle_{\mathbb{R}} = a_{\Gamma,+}[K_\omega g, K_\omega h] - \omega \langle K_\omega g, K_\omega h \rangle_{\Omega_+}.$$

We note that the expression (5.26) is independent of Γ ; in particular the dependence on Γ enters as a constraint on the support of the functions g and h .

Lemma 5.7. *Let $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$. Then $d_{\Gamma,\omega}$ defines a closed sectorial form in $L_2(\Gamma)$. The associated m -sectorial operator is the restriction of $D_{\Gamma,\omega}$ to the operator domain*

$$X_{\Gamma,\omega} := \left\{ g \in \tilde{H}_0^{1/2}(\Gamma) : D_{\Gamma,\omega}g \in L_2(\Gamma) \right\}. \quad (5.27)$$

If ω is real, then the associated operator is self-adjoint.

Proof. For $g \in \tilde{H}_0^{1/2}(\Gamma)$ we use the identity

$$d_{\Gamma,\omega}[g] = d_{\Gamma,\omega}[g, g] = \langle D_\omega g, g \rangle_{\mathbb{R}} = a_{\Gamma,+}[K_\omega g] - \omega \|K_\omega g\|_{L_2(\Omega_+; \mathbb{C}^2)}^2.$$

Let $u := K_\omega g$. The mapping property of the Poisson operator K_ω implies

$$|d_{\Gamma,\omega}[g]| \leq c \|u\|_{H^1(\Omega_+; \mathbb{C}^2)}^2 \leq c_1 \|g\|_{\tilde{H}_0^{1/2}(\Gamma)}^2.$$

Again the mapping properties of the Poisson operator and the trace theorem imply that

$$\begin{aligned} \operatorname{Re}(d_{\Gamma,\omega}[g]) &= \|u\|_{H^1(\Omega_+; \mathbb{C}^2)}^2 - \operatorname{Re}(\omega - 1) \|u\|_{L_2(\Omega_+; \mathbb{C}^2)}^2 \\ &\geq c_1 \|g\|_{\tilde{H}_0^{1/2}(\Gamma)}^2 - c_2 \|g\|_{H^{-1/2}(\mathbb{R})}^2 \geq c_1 \|g\|_{\tilde{H}_0^{1/2}(\Gamma)}^2 - c_3 \|g\|_{L_2(\Gamma)}^2. \end{aligned}$$

Thus, $d_{\Gamma,\omega}$ is closed and accretive. Analogously it follows that $d_{\Gamma,\omega}$ is sectorial. Moreover, a short calculation implies that the associated m -sectorial operator is exactly the restriction of $D_{\Gamma,\omega}$ to $X_{\Gamma,\omega}$. Finally, if $\omega \in \mathbb{R}$ then $d_{\Gamma,\omega}[g] \in \mathbb{R}$, and thus, the associated operator is self-adjoint. \square

As a particular consequence of Lemma 5.7 we obtain that

$$\ker(D_{\Gamma,\omega}) \subseteq X_{\Gamma,\omega}.$$

Since the spectrum of $A_{\Gamma,+}^{(2)}$ is a subset of the real axis we may restrict ourselves the case $\omega \in \mathbb{R}$ and we can apply methods from spectral theory to determine whether zero

is an eigenvalue of $D_{\Gamma,\omega}$ or not. Note that $D[d_{\Gamma,\omega}] = \tilde{H}_0^{1/2}(\Gamma)$ is compactly embedded² into $L_2(\Gamma)$, and thus, the spectrum of the m-sectorial realisation consists of a discrete set of eigenvalues only accumulating at infinity. Moreover, the proof of Lemma 5.7 and [17, Theorem 2.34] imply that the original operator $D_{\Gamma,\omega} : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is a Fredholm operator with zero index.

Naturally the above considerations remain valid if we replace Γ by $\Gamma_\ell := \ell \cdot \Gamma$. We note that the operators $D_{\Gamma_\ell,\omega}$ are each acting in a different Hilbert space for different $\ell > 0$. To obtain a family of operators acting in the same space we introduce the scaling operators

$$T_\ell : L_2(\Gamma) \rightarrow L_2(\Gamma_\ell), \quad (T_\ell g)(x) = \ell^{-1/2} \cdot g(x/\ell) \quad (5.28)$$

and note that the operator T_ℓ bijectively maps $\tilde{H}_0^{1/2}(\Gamma)$ into $\tilde{H}_0^{1/2}(\Gamma_\ell)$. Let

$$\mathcal{Q}(\ell, \omega) : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma), \quad \mathcal{Q}(\ell, \omega) := T_\ell^* D_{\Gamma_\ell,\omega} T_\ell \quad (5.29)$$

with the associated sesquilinear form

$$q(\ell, \omega)[g, h] := d_{\Gamma_\ell,\omega}[T_\ell g, T_\ell h], \quad g, h \in D[q(\ell, \omega)] := \tilde{H}_0^{1/2}(\Gamma). \quad (5.30)$$

Then for $\ell > 0$ and $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ we have

$$\dim \ker(A_{\Gamma_\ell}^{(2)} - \omega) = \dim \ker(D_{\Gamma_\ell,\omega}) = \dim \ker(\mathcal{Q}(\ell, \omega)).$$

From (5.26) we obtain for $g, h \in \tilde{H}_0^{1/2}(\Gamma)$

$$q(\ell, \omega)[g, h] = d_{\Gamma_\ell,\omega}[T_\ell g, T_\ell h] = \ell \int_{\mathbb{R}} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi = \int_{\mathbb{R}} m_\omega(\xi/\ell) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi.$$

Next we describe the behaviour of the form $q(\ell, \omega)$ as $\ell \rightarrow 0$ and $\omega \rightarrow \Lambda$. This asymptotic expansion will represent the principal tool for the proof of Theorem 3.1. Let $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$,

$$\langle \mathcal{Q}_0 g, h \rangle_\Gamma := q_0[g, h] := \int_{\mathbb{R}} |\xi| \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi. \quad (5.31)$$

We note that $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is a Fredholm operator with Fredholm index 0, which follows from [17, Theorem 2.34]. Hence, \mathcal{Q}_0 is invertible since it has trivial kernel. Indeed, the identity $\mathcal{Q}_0 g = 0$ implies that

$$0 = \langle \mathcal{Q}_0 g, g \rangle_\Gamma = \int_{\mathbb{R}} |\xi| \cdot |\hat{g}(\xi)|^2 \, d\xi,$$

and thus, $g = 0$. Furthermore, we denote by P_\pm the projection onto the subspace in $L_2(\Gamma)$ spanned by the functions

$$\Phi_\pm(x_1) := e^{\pm i\kappa x_1} \quad (5.32)$$

and let $\psi_{\pm\kappa} = (\psi_{\pm\kappa,1}, \psi_{\pm\kappa,2})^T \in L_2(I_+; \mathbb{C}^2)$ be chosen such that

$$A_{\partial^+}^{(2)}(\pm\kappa)\psi_{\pm\kappa} = \Lambda\psi_{\pm\kappa} \quad \text{and} \quad \|\psi_{\pm\kappa}\|_{L_2(I_+; \mathbb{C}^2)} = 1, \quad (5.33)$$

²cf. [17, Theorem 3.27].

cf. also Formula (4.11), where a non-normalised eigenfunction for the unitarily equivalent operator $A_{\varnothing}^{(2)}$ is given.

Theorem 5.8. *There exists $\ell_0 > 0$ and $\varepsilon > 0$ such that for all $\ell \in (0, \ell_0)$ and $|\omega - \Lambda| < \varepsilon$ the following expansion holds true*

$$\mathcal{Q}(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 - \frac{4|\Gamma| \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} T_\ell^* (P_+ + P_-) T_\ell + R(\ell, \omega). \quad (5.34)$$

Here $|\Gamma|$ is the Lebesgue measure of Γ and the remainder satisfies the following estimate

$$\sup\{\|R(\ell, \omega)\|_{\mathcal{L}(L_2(\Gamma))} : \ell \in (0, \ell_0) \wedge |\omega - \Lambda| < \varepsilon\} < \infty.$$

The next section is devoted to the proof of Theorem 5.8.

5.3. The proof of Theorem 5.8. For the proof we use the perturbation formula for the Dirichlet-to-Neumann operator D_ω in Theorem 5.5 (4), which we apply for $\eta = 0$. We obtain

$$D_\omega = D_0 - \omega K_0 (I + \omega (A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0^*.$$

Thus, for $g, h \in \tilde{H}_0^{1/2}(\Gamma)$ we have

$$\begin{aligned} \langle \mathcal{Q}(\ell, \omega) g, h \rangle_\Gamma &= q(\ell, \omega)[g, h] = d_{\Gamma_\ell, \omega}[T_\ell g, T_\ell h] = \langle D_\omega T_\ell g, T_\ell h \rangle_\mathbb{R} \\ &= \langle D_0 T_\ell g, T_\ell h \rangle_\mathbb{R} - \omega \langle (I + \omega (A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 T_\ell g, K_0 T_\ell h \rangle_{\Omega_+} \\ &= q(\ell, 0)[g, h] - \omega \langle (I + \omega (A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 T_\ell g, K_0 T_\ell h \rangle_{\Omega_+}. \end{aligned}$$

Recall that

$$q(\ell, 0)[g, h] = \int_{\mathbb{R}} m_0(\xi/\ell) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi$$

with

$$m_0(\xi) = \xi \cdot \frac{\cosh(\pi\xi) - 1 - \frac{\pi^2 \xi^2}{2}}{\sinh(\pi\xi) + \pi\xi}.$$

Using the estimate $m_0(\xi) = |\xi| + \mathcal{O}(1)$ we obtain

$$q(\ell, 0)[g, h] = \frac{1}{\ell} \int_{\mathbb{R}} |\xi| \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi + \langle \tilde{R}(\ell) g, h \rangle_\Gamma = \frac{1}{\ell} \langle \mathcal{Q}_0 g, h \rangle_\Gamma + \langle \tilde{R}(\ell) g, h \rangle_\Gamma,$$

where

$$\sup\{\|\tilde{R}(\ell)\|_{\mathcal{L}(L_2(\Gamma))} : \ell \in (0, 1)\} < \infty.$$

Now we treat the resolvent term. To this end we need to understand the behaviour of the solutions of the Rayleigh-Lamb equation (4.10) as $\omega \rightarrow \Lambda$.

Lemma 5.9. *Let $\beta = \sqrt{\omega - \xi^2}$, $\gamma = \sqrt{\frac{\omega}{2} - \xi^2}$. There exist $\Theta > 0$ and $\varepsilon > 0$ such that for all $|\omega - \Lambda| < \varepsilon$ the Rayleigh-Lamb equation*

$$\frac{\sin(\frac{\beta\pi}{2})}{\beta} \cos\left(\gamma\frac{\pi}{2}\right) \gamma^2 + \cos\left(\beta\frac{\pi}{2}\right) \frac{\sin(\frac{\gamma\pi}{2})}{\gamma} \xi^2 = 0$$

has exactly four solutions in the infinite strip $\mathbb{R} + i[-\Theta, \Theta]$. For $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$, two of these solutions have strictly positive imaginary part and two of them have strictly

negative imaginary part. There exists a holomorphic function H with $H(0) = \varkappa$ and $H'(0) = (\zeta_1''(\varkappa)/2)^{-1/2}$ such that the solutions with positive imaginary part are given by

$$\xi_{1+}(\omega) = -H(-i\sqrt{\Lambda - \omega}) \quad \text{and} \quad \xi_{2+}(\omega) = H(i\sqrt{\Lambda - \omega}).$$

We recall that $\zeta_1(\xi)$ is the first eigenvalue of the operators $A_{\varnothing}^{(2)}(\xi)$ and $A_{\varnothing+}^{(2)}(\xi)$ for $\xi \in \mathbb{R}$. Moreover, we have

$$\zeta_1(\pm\varkappa) = \Lambda = \min\{\zeta_1(\xi) : \xi \in \mathbb{R}\}.$$

Proof. Let

$$\Psi(\xi, \omega) := \frac{\sin(\beta\frac{\pi}{2})}{\beta} \cos\left(\gamma\frac{\pi}{2}\right) \gamma^2 + \cos\left(\beta\frac{\pi}{2}\right) \frac{\sin(\gamma\frac{\pi}{2})}{\gamma} \xi^2$$

be the left-hand side of the Rayleigh-Lamb equation, where as before

$$\beta = \sqrt{\omega - \xi^2} \quad \text{and} \quad \gamma = \sqrt{\frac{\omega}{2} - \xi^2}.$$

Expanding the sine and cosine functions into their power series shows that only powers of the square root function with even exponent are present, and thus, Ψ is holomorphic in both variables $(\xi, \omega) \in \mathbb{C}^2$. We note that for values $\xi \in \mathbb{R}$ and $\omega \in \mathbb{C}$ with $\Psi(\xi, \omega) = 0$ we necessarily have $\zeta_k(\xi) = \omega$ for some $k \in \mathbb{N}$. In the case $\omega = \Lambda$ the function $\mathbb{R} \ni \xi \mapsto \Psi(\xi, \Lambda)$ has exactly two zeros $\pm\varkappa$, cf. Section 4 or [3]. Each of them has multiplicity 2, i.e., we have

$$\partial_{\xi}\Psi(\pm\varkappa, \Lambda) = 0 \quad \text{and} \quad \partial_{\xi}^2\Psi(\pm\varkappa, \Lambda) \neq 0. \quad (5.35)$$

The argument principle for holomorphic functions implies that there exist two constants $\Theta > 0$ and $\delta > 0$ such that $\Psi(\cdot, \omega)$ has exactly 4 zeros counted with multiplicities in the infinite strip $\mathbb{R} + i[-\Theta, \Theta]$ for $|\omega - \Lambda| < \varepsilon$. Two of these zeros are located near \varkappa and the others are near $-\varkappa$. Note that we used that $|\Psi(\xi, \omega)|$ tends to infinity as $\text{Re}(\xi) \rightarrow \pm\infty$, locally uniform in ω .

For the sake of simplicity we consider for the rest of the proof only those zeros near \varkappa . We note that the function ζ_1 is chosen such that $\Psi(\xi, \zeta_1(\xi)) = 0$ for $\xi \in \mathbb{R}$. Moreover, we have $\partial_{\omega}\Psi(\varkappa, \Lambda) \neq 0$. This follows by a numerical calculation, which can be made rigorous by approximating the corresponding power series, cf. also the considerations in [3]. Thus, there exists neighbourhoods U_{\varkappa} of \varkappa and V_{Λ} of Λ so that for all $(\xi, \omega) \in U_{\varkappa} \times V_{\Lambda}$ the identity $\Psi(\xi, \omega) = 0$ holds true if and only if $\omega = \zeta_1(\xi)$. Note that $\zeta_1'(\varkappa) = 0$ since \varkappa is a global minimum and $\zeta_1''(\varkappa) \neq 0$. As ζ_1 is real analytic there exists a neighbourhood V_0 around 0 and an invertible analytic function $G : U_{\varkappa} \rightarrow V_0$ such that

$$\zeta_1(\xi) = \Lambda + G(\xi)^2, \quad \xi \in U_{\varkappa}.$$

Setting $H := G^{-1}$ we observe that the two zeros of $\Psi(\cdot, \omega)$ near \varkappa are given by

$$G^{-1}(\pm i\sqrt{\omega - \Lambda}) = H(\pm i\sqrt{\omega - \Lambda}).$$

Note that we may choose G such that $G'(\varkappa) = (\zeta_1''(\varkappa)/2)^{1/2}$, and thus, $H'(0) = (\zeta_1''(\varkappa)/2)^{-1/2} > 0$. The Taylor expansion of H shows that one zero has strictly positive imaginary part, the other strictly negative imaginary part. \square

Now we may give an estimate for the resolvent term. Let $g, h \in \tilde{H}_0^{1/2}(\Gamma)$. For ease of notation we put $g_\ell := T_\ell g$ and $h_\ell := T_\ell h$. Then

$$\begin{aligned} & \omega \langle (I + \omega(A_{\varnothing+}^{(2)} - \omega)^{-1})K_0 T_\ell g, K_0 T_\ell h \rangle_{\Omega+} \\ &= \omega \int_{\mathbb{R}} \langle (I + \omega(A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1})K_0(\xi) \hat{g}_\ell(\xi), K_0(\xi) \hat{h}_\ell(\xi) \rangle_{I+}. \end{aligned} \quad (5.36)$$

In what follows we use that $K_0(\cdot) : \mathbb{C} \rightarrow H^2(I_+; \mathbb{C}^2) \cap h_{2+}$ is a finitely meromorphic³ function with singularities, which are located at most at those points $\xi \in \mathbb{C}$ which solve the Rayleigh-Lamb equation $\Psi(\xi, 0) = 0$. In particular we may choose $\Theta > 0$ and $\varepsilon > 0$ such that $K_0(\cdot)$ is holomorphic in $\mathbb{R} + i[-2\Theta, 2\Theta]$ and such that for all $|\omega - \Lambda| < \varepsilon$ the Rayleigh-Lamb equation has exactly four solutions in the infinite strip $\mathbb{R} + i[-2\Theta, 2\Theta]$. Let

$$F_\omega(\xi) := \langle (I + \omega(A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1})K_0(\xi) \hat{g}_\ell(\xi), K_0(\bar{\xi}) \hat{h}_\ell(\bar{\xi}) \rangle_{I+}$$

be the integrand in (5.36). Then F_ω may be extended to a meromorphic function on the strip $\mathbb{R} + i[-2\Theta, 2\Theta]$ since the functions $\hat{g}_\ell, \hat{h}_\ell$ have compact support, and thus, $\hat{g}_\ell, \hat{h}_\ell$ may be extended to holomorphic functions on all of \mathbb{C} . We have

$$\int_{\mathbb{R}} F_\omega(\xi) \, d\xi = \int_{-\varkappa-}^{-\varkappa+\delta} F_\omega(\xi) \, d\xi + \int_{\varkappa-\delta}^{\varkappa+\delta} F_\omega(\xi) \, d\xi + \left(\int_{-\infty}^{-\varkappa-\delta} + \int_{-\varkappa+\delta}^{\varkappa-\delta} + \int_{\varkappa+\delta}^{\infty} \right) F_\omega(\xi) \, d\xi$$

for some $\delta > 0$. Note that

$$\begin{aligned} \left| \int_{-\infty}^{-\varkappa-\delta} F_\omega(\xi) \, d\xi \right| &\leq \sup_{\xi < -\varkappa-\delta} \|(I + \omega(A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1}) \cdot \| \cdot \|K_0 g_\ell\|_{L_2(\Omega_+; \mathbb{C}^2)}^2 \|K_0 h_\ell\|_{L_2(\Omega_+; \mathbb{C}^2)}^2 \\ &\leq C \|g\|_{L_2(\Gamma)} \cdot \|h\|_{L_2(\Gamma)} = \mathcal{O}(1), \end{aligned}$$

since the resolvent may be estimated by the distance of ω to the spectrum of $A_{\varnothing+}^{(2)}(\xi)$ and

$$\int_{\mathbb{R}} \|K_0(\xi) \hat{g}_\ell(\xi)\|_{L_2(I_+)}^2 \, d\xi = \|K_0 g_\ell\|_{L_2(\Omega_+; \mathbb{C}^2)}^2$$

In the same way we may treat the integrals $\int_{-\varkappa+\delta}^{\varkappa-\delta} F_\omega(\xi) \, d\xi$ and $\int_{\varkappa+\delta}^{\infty} F_\omega(\xi) \, d\xi$. Thus,

$$\left(\int_{-\infty}^{-\varkappa-\delta} + \int_{-\varkappa+\delta}^{\varkappa-\delta} + \int_{\varkappa+\delta}^{\infty} \right) F_\omega(\xi) \, d\xi = \mathcal{O}(1).$$

To estimate the remaining integrals we consider the following expansion of the resolvent

$$\begin{aligned} I + \omega(A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1} &= \sum_{k=1}^{\infty} \left(1 + \frac{\omega}{\zeta_k(\xi) - \omega} \right) P_k(\xi) \\ &= \frac{\zeta_1(\xi)}{\zeta_1(\xi) - \omega} P_1(\xi) + \sum_{k=2}^{\infty} \left(1 + \frac{\omega}{\zeta_k(\xi) - \omega} \right) P_k(\xi), \end{aligned}$$

³For the definition of finitely meromorphic functions we refer to [20].

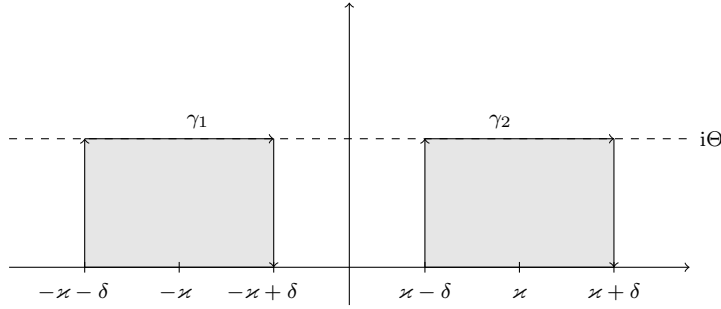
where $P_k(\xi)$ is the projection onto the eigenspaces $\ker(A_{\emptyset+}^{(2)}(\xi) - \zeta_k(\xi))$. Note that

$$\left\| \sum_{k=2}^{\infty} \left(1 + \frac{\omega}{\zeta_k(\xi) - \omega} \right) P_k(\xi) \right\|_{\mathcal{L}(L_2(I_+))} \leq 1 + \frac{\omega}{\min \{ \zeta_2(\xi) : \xi \in \mathbb{R} \} - \omega} \leq \tilde{c},$$

for $|\omega - \Lambda| < \varepsilon$. Thus,

$$\int_{\pm\kappa-\delta}^{\pm\kappa+\delta} F_{\omega}(\xi) \, d\xi = \int_{\pm\kappa-\delta}^{\pm\kappa+\delta} \frac{1}{\zeta_1(\xi) - \omega} \langle P_1(\xi) K_0(\xi) \hat{g}_{\ell}(\xi), K_0(\xi) \hat{h}_{\ell}(\xi) \rangle_{I_+} \, d\xi + \mathcal{O}(1).$$

Now we choose two paths γ_j , $j = 1, 2$, in the complex plane which run around the boundaries of the following rectangles except for the two line segments on the real axis:



Applying Lemma 5.9 we may assume that for $|\omega - \Lambda| < \varepsilon$ the function $(\zeta_1(\cdot) - \omega)^{-1}$ has exactly one singularity in each rectangle. By Lemma 5.9 the singularities are given by

$$\xi_{1+}(\omega) = -H(-i\sqrt{\Lambda - \omega}) \quad \text{and} \quad \xi_{2+}(\omega) = H(i\sqrt{\Lambda - \omega})$$

for some holomorphic function H satisfying $H(0) = \kappa$ and $H'(0) = (\zeta_1''(\kappa)/2)^{-1/2}$. Note that $H(-\eta) = -H(\eta)$. Since $\xi \mapsto P_1(\xi)$ depends holomorphically on ξ the residue theorem implies

$$\begin{aligned} & \left(\int_{-\kappa-\varepsilon}^{-\kappa+\varepsilon} + \int_{\kappa-\varepsilon}^{\kappa+\varepsilon} \right) F_{\omega}(\xi) \, d\xi \\ &= \left(\int_{\gamma_1} + \int_{\gamma_2} \right) \frac{\zeta_1(\xi)}{\zeta_1(\xi) - \omega} \langle P_1(\xi) K_0(\xi) \hat{g}_{\ell}(\xi), K_0(\bar{\xi}) \hat{h}_{\ell}(\bar{\xi}) \rangle_{I_+} \, d\xi \\ & \quad + 2\pi i [\text{Res}_{\xi=\xi_{1+}(\omega)} + \text{Res}_{\xi=\xi_{2+}(\omega)}] \left(\frac{\zeta_1(\xi) \cdot \langle P_1(\xi) K_0(\xi) \hat{g}_{\ell}(\xi), K_0(\bar{\xi}) \hat{h}_{\ell}(\bar{\xi}) \rangle_{I_+}}{\zeta_1(\xi) - \omega} \right) \\ & \quad + \mathcal{O}(1). \end{aligned}$$

Note that

$$\int_{\gamma_j} \frac{\zeta_1(\xi)}{\zeta_1(\xi) - \omega} \langle P_1(\xi) K_0(\xi) \hat{g}_{\ell}(\xi), K_0(\bar{\xi}) \hat{h}_{\ell}(\bar{\xi}) \rangle_{I_+} \, d\xi = \mathcal{O}(1) \quad \text{for } j = 1, 2.$$

For the residues we have

$$\begin{aligned} \operatorname{Res}_{\xi=\xi_{1+}(\omega)} \frac{1}{\zeta_1(\xi) - \omega} &= \operatorname{Res}_{\xi=\xi_{2+}(\omega)} \frac{1}{\zeta_1(\xi) - \omega} = \frac{1}{\zeta_1'(\xi)} \Big|_{\xi=\xi_{2+}(\omega)} \\ &= \frac{1}{\zeta_1'(H(i\sqrt{\Lambda - \omega}))} = \sum_{k=-1}^{\infty} c_k (\Lambda - \omega)^{k/2} \end{aligned}$$

for coefficients c_k . The singular term in the Laurent series is given by

$$c_{-1} = \frac{1}{i \zeta_1''(\varkappa) H'(0)} = \frac{1}{i \sqrt{2\zeta_1''(\varkappa)}} \neq 0.$$

Expanding the term $\langle P_1(\xi) K_0(\xi) \hat{g}_\ell(\xi), K_0(\bar{\xi}) \hat{h}_\ell(\bar{\xi}) \rangle$ into a Taylor series at $\xi = \pm \varkappa$ we obtain

$$\begin{aligned} \omega \int_{\mathbb{R}} F_\omega(\xi) \, d\xi &= \frac{2\pi \cdot \omega^2}{\sqrt{2\zeta_1''(\varkappa)} \cdot \sqrt{\Lambda - \omega}} \sum_{\diamond=\pm\varkappa} \langle P_0(\diamond) K_0(\diamond) \hat{g}_\ell(\diamond), K_0(\diamond) \hat{h}_\ell(\diamond) \rangle_{I_+} + \mathcal{O}(1) \\ &= \frac{2\pi \cdot \Lambda^2}{\sqrt{2\zeta_1''(\varkappa)} \cdot \sqrt{\Lambda - \omega}} \sum_{\diamond=\pm\varkappa} \langle P_0(\diamond) K_0(\diamond) \hat{g}_\ell(\diamond), K_0(\diamond) \hat{h}_\ell(\diamond) \rangle_{I_+} + \mathcal{O}(1). \end{aligned}$$

For $\diamond = \pm \varkappa$ we obtain

$$\Lambda^2 \langle P_1(\diamond) K_0(\diamond) \hat{g}_\ell(\diamond), K_0(\diamond) \hat{h}_\ell(\diamond) \rangle_{I_+} = \Lambda^2 \langle K_0(\diamond) g_\ell(\diamond), \psi_\diamond \rangle_{I_+} \langle \psi_\diamond, K_0(\diamond) \hat{h}_\ell(\diamond) \rangle_{I_+},$$

and

$$\Lambda \langle K_0(\diamond) g_\ell(\diamond), \psi_\diamond \rangle_{I_+} = \langle K_0(\diamond) g_\ell(\diamond), A_{\diamond+}^{(2)}(\diamond) \psi_\diamond \rangle_{I_+} = -2\overline{\partial_n \psi_{\diamond,2}(0)} \cdot \hat{g}_\ell(\diamond).$$

Note that

$$\begin{aligned} \hat{g}_\ell(\diamond) \cdot \overline{\hat{h}_\ell(\diamond)} &= \frac{1}{2\pi} \left(\int_{\Gamma} e^{-i\diamond x} g_\ell(x) \, dx \right) \overline{\left(\int_{\Gamma} e^{-i\diamond x} h_\ell(x) \, dx \right)} \\ &= \frac{1}{2\pi} \langle g_\ell, \Phi_\pm \rangle_{\Gamma} \cdot \langle \Phi_\pm, h_\ell \rangle_{\Gamma} = \frac{|\Gamma|}{2\pi} \langle P_\pm g_\ell, h_\ell \rangle_{I_+}, \end{aligned}$$

where $|\Gamma|$ is the Lebesgue measure of Γ . Finally we obtain

$$\Lambda^2 \langle P_1(\diamond) K_0(\diamond) \hat{h}_\ell(\diamond), K_0(\diamond) \hat{h}_\ell(\diamond) \rangle_{I_+} = \frac{2|\Gamma|}{\pi} \cdot |\partial_n \psi_{\diamond,2}(0)|^2 \cdot \langle P_\pm T_\ell g, T_\ell h \rangle_{I_+},$$

which proves Theorem 5.8 since $\psi_{\varkappa,2} = \psi_{-\varkappa,2}$.

5.4. The proof of Theorem 3.1. From Theorem 5.8 we obtain

$$\mathcal{Q}(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 + \frac{4|\Gamma| \cdot |\partial_2 \psi_{\varkappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} T_\ell^* (P_+ + P_-) T_\ell + R(\ell, \omega) \quad (5.37)$$

with the following estimate on the remainder

$$\sup\{\|R(\ell, \omega)\|_{\mathcal{L}(L_2(\Gamma))} : \ell \in (0, \ell_0) \wedge |\omega - \Lambda| < \varepsilon\} < \infty. \quad (5.38)$$

Remark. Using a similar argumentation as in Theorem 5.8 it follows that for every compact set $K \subseteq \mathbb{C} \setminus [\Lambda, \infty)$ there exists $\ell_0 = \ell_0(\Gamma, K)$ such that

$$\mathcal{Q}(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 + \tilde{R}(\ell, \omega),$$

and the remainder satisfies

$$\sup\{\|\tilde{R}(\ell, \omega)\|_{\mathcal{L}(L_2(\Gamma))} : \omega \in K \wedge \ell \in (0, \ell_0)\} < \infty.$$

Recalling that the operator \mathcal{Q}_0 is invertible, we obtain

$$\mathcal{Q}(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 \left(I + \ell \mathcal{Q}_0^{-1} \tilde{R}(\ell, \omega) \right).$$

Choosing $\ell > 0$ sufficiently small implies that $\mathcal{Q}(\ell, \omega)$ is invertible for all $\omega \in K$ and $\ell \in (0, \ell_0)$. In particular, 0 cannot be an eigenvalue of $\mathcal{Q}(\ell, \omega)$. As a consequence the discrete eigenvalues of the operator $A_{\Gamma_\ell}^{(2)}$ converge to Λ as $\ell \rightarrow 0$.

Now we use the symmetry of the problem with respect to the axis $x_1 = 0$. We set

$$L_{2,s}(\Gamma) := \{g \in L_2(\Gamma) : g(x_1) = g(-x_1)\}, \quad (5.39)$$

$$L_{2,as}(\Gamma) := \{g \in L_2(\Gamma) : g(x_1) = -g(-x_1)\} \quad (5.40)$$

with projections P_s and P_{as} . Recall that $\Gamma = -\Gamma$. Since in we are now considering a mixed problem on the upper half-strip there will be no risk of confusion with the spaces $\mathcal{H}^{(s)}$ and $\mathcal{H}^{(as)}$ introduced before. A simple calculation shows that the forms $q(\ell, \omega)$ and q_0 decompose as follows

$$q(\ell, \omega) = q^{(s)}(\ell, \omega) \oplus q^{(as)}(\ell, \omega) \quad \text{and} \quad q_0 = q_0^{(s)} \oplus q_0^{(as)},$$

where $q^{(\dagger)}(\ell, \omega)$ and $q_0^{(\dagger)}$ act in $L_{2,\dagger}(\Gamma)$ for $\dagger \in \{s, as\}$. Thus,

$$\mathcal{Q}(\ell, \omega) = P_s^* \mathcal{Q}(\ell, \omega) P_s + P_{as}^* \mathcal{Q}(\ell, \omega) P_{as}$$

and Theorem 5.8 implies

$$P_\dagger^* \mathcal{Q}(\ell, \omega) P_\dagger = \frac{1}{\ell} P_\dagger^* \mathcal{Q}_0 P_\dagger - \frac{4|\Gamma| \cdot |\partial_2 \psi_{\varkappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} P_\dagger^* T_\ell^* (P_+ + P_-) T_\ell P_\dagger + P_\dagger^* R(\ell, \omega) P_\dagger$$

for $\dagger \in \{s, as\}$. A short calculation shows that for fixed $\dagger \in \{s, as\}$ the operators $P_\dagger^* T_\ell^* P_+ T_\ell P_\dagger$ and $P_\dagger^* T_\ell^* P_- T_\ell P_\dagger$ coincide. Indeed, we have

$$P_s^* T_\ell^* P_\pm T_\ell P_s = \frac{\ell}{|\Gamma|} \langle \cdot, \Phi_\ell^{(s)} \rangle_\Gamma \cdot \Phi_\ell^{(s)} \quad \text{and} \quad P_{as}^* T_\ell^* P_\pm T_\ell P_{as} = \frac{\ell}{|\Gamma|} \langle \cdot, \Phi_\ell^{(as)} \rangle_\Gamma \cdot \Phi_\ell^{(as)},$$

where

$$\Phi_\ell^{(s)}(x_1) := \cos(\varkappa \ell x_1) \quad \text{and} \quad \Phi_\ell^{(as)}(x_1) := \sin(\varkappa \ell x_1).$$

Thus,

$$P_\dagger^* \mathcal{Q}(\ell, \omega) P_\dagger = \frac{1}{\ell} P_\dagger^* \mathcal{Q}_0 P_\dagger - \frac{8\ell \cdot |\partial_2 \psi_{\varkappa,2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle \cdot, \Phi_\ell^{(\dagger)} \rangle_\Gamma \cdot \Phi_\ell^{(\dagger)} + P_\dagger^* R(\ell, \omega) P_\dagger.$$

In particular $P_{\dagger}^* T_{\ell}^* (P_+ + P_-) T_{\ell} P_{\dagger}$ are rank-one operators for $\dagger \in \{\text{s}, \text{as}\}$. To prove Theorem 3.1 we consider for real ω not only the kernel of the operator $\mathcal{Q}(\ell, \omega)$, but more generally the discrete eigenvalues of the self-adjoint realisation of $P_{\dagger}^* \mathcal{Q}(\ell, \omega) P_{\dagger}$ in $L_{2,\dagger}(\Gamma)$ for $\dagger \in \{\text{s}, \text{as}\}$. For $\ell > 0$ and $\omega \in \mathbb{R} \setminus [\Lambda, \infty)$ we denote these eigenvalues (counted with multiplicities) by

$$\mu_1^{(\dagger)}(\ell, \omega) \leq \mu_2^{(\dagger)}(\ell, \omega) \leq \dots$$

Lemma 5.10. *Let $\ell_0 > 0$ and $\varepsilon > 0$ be chosen as in Theorem 5.8. For $\dagger \in \{\text{s}, \text{as}\}$ the following assertions hold true:*

- (1) *For $\ell > 0$ the function $\mu_1^{(\dagger)}(\ell, \cdot)$ is strictly decreasing in the interval $(-\infty, \Lambda)$.*
- (2) *For fixed $\ell \in (0, \ell_0)$ we have $\mu_1^{(\dagger)}(\ell, \omega) \rightarrow -\infty$ as $\omega \rightarrow \Lambda$.*
- (3) *For fixed $\omega \in (\Lambda - \varepsilon, \Lambda)$ we have $\mu_1^{(\dagger)}(\ell, \omega) \rightarrow \infty$ as $\ell \rightarrow 0$.*
- (4) *There exists $\tilde{\ell}_0 > 0$ such that for all $\tilde{\ell} \in (0, \ell_0)$ and for all $|\omega - \Lambda| < \varepsilon$ we have $\mu_2^{(\dagger)}(\tilde{\ell}, \omega) > 0$.*

Proof. For the proof of (1) we use the decomposition of the Dirichlet-to-Neumann operator D_{ω} given in Theorem 5.5. For $\omega_1, \omega_2 < \Lambda$ and $g \in \tilde{H}_0^{1/2}(\Gamma)$ we have

$$\begin{aligned} q(\ell, \omega_1) - q(\ell, \omega_2)[g] &= -\omega_1 \langle (I + \omega_1(A_{\varnothing+}^{(2)} - \omega_1)^{-1}) K_0 T_{\ell} g, K_0 T_{\ell} g \rangle_{\Omega_+} \\ &\quad + \omega_2 \langle (I + \omega_2(A_{\varnothing+}^{(2)} - \omega_2)^{-1}) K_0 T_{\ell} g, K_0 T_{\ell} g \rangle_{\Omega_+} \\ &= \int_{[\Lambda, \infty)} \left(-\frac{\nu \omega_1}{\nu - \omega_1} + \frac{\nu \omega_2}{\nu - \omega_2} \right) d\langle E(\nu) K_0 T_{\ell} g, K_0 T_{\ell} g \rangle_{\Omega_+}, \end{aligned}$$

where $E(\nu)$ is the spectral resolution of the operator $A_{\varnothing+}^{(2)}$. A short calculation shows that the above integrand is strictly positive if $\omega_1 < \omega_2 < \nu$. Now the first assertion follows from the min-max principle for self-adjoint operator applied to the form $q^{(\dagger)}(\ell, \omega)$ for $\dagger \in \{\text{s}, \text{as}\}$.

Here and subsequently we fix $\dagger \in \{\text{s}, \text{as}\}$. To prove assertion (2) we use Theorem 5.8 and the min-max principle for self-adjoint operators. Let $\ell \in (0, \ell_0)$. For $|\omega - \Lambda| < \varepsilon$ we have $\mu_1^{(\dagger)}(\ell, \omega) \leq q^{(\dagger)}(\ell, \omega)[g_0]$ for every $g_0 \in \tilde{H}_0^{1/2}(\Gamma) \cap L_{2,\dagger}(\Gamma)$ with $\|g_0\|_{L_2(\Gamma)} = 1$. Choosing g_0 such that

$$\langle T_{\ell}^* (P_+ + P_-) T_{\ell} g_0, g_0 \rangle_{\Gamma} \neq 0,$$

we obtain from Theorem 5.8

$$\mu_1^{(\dagger)}(\ell, \omega) \leq \frac{1}{\ell} \langle \mathcal{Q}_0 g_0, g_0 \rangle_{\Gamma} - \frac{4|\Gamma| \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \cdot \langle T_{\ell}^* (P_+ + P_-) T_{\ell} g_0, g_0 \rangle_{\Gamma} + C_1,$$

which tends to $-\infty$ as $\omega \rightarrow \Lambda$. Here $C_1 := \sup\{\|R(\ell, \omega)\|_{\mathcal{L}(L_2(\Gamma))} : \ell \in (0, \ell_0) \wedge |\omega - \Lambda| < \varepsilon\}$. This proves the second assertion. To deduce (3) we recall that \mathcal{Q}_0 is invertible and we have $q_0[g] = \langle \mathcal{Q}_0 g, g \rangle_{\Gamma} \geq 0$ for all $g \in \tilde{H}_0^{1/2}(\Gamma)$. Moreover, there exists $\mu_* > 0$ such that

$$\langle \mathcal{Q}_0 g, g \rangle_{\Gamma} = q_0[g] \geq \mu_* \|g\|_{L_2(\Gamma)}^2, \quad g \in \tilde{H}_0^{1/2}(\Gamma).$$

This follows since the spectrum of the self-adjoint realisation of \mathcal{Q}_0 is purely discrete and from the fact that 0 cannot be an eigenvalue. Hence, for fixed $\omega \in \mathbb{R} \setminus [\Lambda, \infty)$, $|\omega - \Lambda| < \varepsilon$ we have

$$\mu_1^{(\dagger)}(\ell, \omega) = \inf\{q^{(\dagger)}(\ell, \omega)[g] : g \in \tilde{H}_0^{1/2}(\Gamma) \cap L_{2,\dagger}(\Gamma) \wedge \|g\|_{L_2(\Gamma)} = 1\} \geq \frac{\mu_*}{\ell} - C_1,$$

which tends to ∞ as $\ell \rightarrow 0$. This proves (3). Assertion (4) follows if we prove that the form $q^{(\dagger)}(\ell, \omega)$ is positive on a subset of codimension 1. Choose $g \in \tilde{H}_0^{1/2}(\Gamma) \cap L_{2,\dagger}(\Gamma)$, $\|g\|_{L_2(\Gamma)} = 1$, orthogonal to the function $\Phi_\ell^{(\dagger)}$. Then

$$q^{(\dagger)}(\ell, \omega)[g] = \frac{1}{\ell} q_0[g] + \langle R(\ell, \omega)g, g \rangle_\Gamma \geq \frac{\mu_*}{\ell} - C_1 > 0$$

for $0 < \ell < \tilde{\ell}_0 := \min\{1, \mu_*/C_1\}$ and $|\omega - \Lambda| < \varepsilon$. This concludes the proof of Lemma 5.10. \square

Lemma 5.11. *There exists $\ell_0 = \ell_0(\Gamma) > 0$ such that for all $\ell \in (0, \ell_0)$ the operator $A_{\Gamma_\ell}^{(2)}$ has exactly two eigenvalues below its essential spectrum $[\Lambda, \infty)$.*

Proof. The assertion follows if we show for some $\ell_0 > 0$ that for all $\ell \in (0, \ell_0)$ there exists unique $\lambda_1(\ell), \lambda_2(\ell) \in (-\infty, \Lambda)$ such that $\mu_1^{(s)}(\ell, \lambda_1(\ell)) = 0 = \mu_1^{(as)}(\ell, \lambda_2(\ell))$. Fix $\dagger \in \{s, as\}$ and let $\varepsilon > 0$ be chosen as in Theorem 5.8 and Lemma 5.10. Using the remark at the beginning of Section 5.4 we choose $\ell_0 > 0$ such that $\inf \sigma(A_{\Gamma_\ell}^{(2)}) \geq \Lambda - \varepsilon$ and $\mu_2^{(\dagger)}(\ell, \omega) > 0$ for all $\ell \in (0, \ell_0)$ and $\omega \in (\Lambda - \varepsilon, \Lambda)$. Let $\ell \in (0, \ell_0)$. If ω is chosen such that $\mu_1^{(\dagger)}(\ell, \omega) = 0$, then Lemma 5.10 (1) implies for $\omega_1 < \omega < \omega_2 < \Lambda$

$$\mu_1^{(\dagger)}(\ell, \omega_1) < \mu_1^{(\dagger)}(\ell, \omega) = 0 < \mu_1^{(\dagger)}(\ell, \omega_2).$$

As a consequence $A_{\Gamma_\ell}^{(2)}$ may have at most two discrete eigenvalues for $\ell \in (0, \ell_0)$.

For the sake of completeness we shall also prove the existence of the eigenvalues. Using Lemma 5.10 (3) we may assume that $\mu_1^{(\dagger)}(\ell, \Lambda - \varepsilon/2) > 0$ for all $\ell \in (0, \ell_0)$. Fix $\ell \in (0, \ell_0)$. Since $\mu_1^{(\dagger)}(\ell, \omega) \rightarrow -\infty$ as $\omega \rightarrow \Lambda$ and $\mu_1^{(\dagger)}(\ell, \omega)$ depends continuously on ω it follows that there exists $\tilde{\omega} = \tilde{\omega}(\ell) \in (\Lambda - \varepsilon/2, \Lambda)$ such that $\mu_1^{(\dagger)}(\ell, \tilde{\omega}) = 0$. Thus, $\tilde{\omega} \in \sigma_d(A_{\Gamma_\ell}^{(2)})$. \square

Remark. Another method of proof for Lemma 5.11 may be based on a variant of operator-valued Rouché's theorem, cf. e.g. [7, 20].

The proof of the asymptotic formula for the eigenvalue of $A_{\Gamma_\ell}^{(2)}$ is based on the Birman-Schwinger principle. Using the estimate (5.38) we may choose $\ell_0 > 0$ such that the operator $\mathcal{Q}_0 + \ell R(\ell, \omega)$ is invertible for all $\ell \in (0, \ell_0)$ and $\omega \in (\Lambda - \varepsilon, \Lambda)$.

Lemma 5.12 (Birman-Schwinger principle for rank-one perturbations). *We denote by $T : D(T) \subseteq H \rightarrow H$ a self-adjoint operator acting in a Hilbert space H with $0 \notin \sigma(T)$. Let $V \in \mathcal{L}(H)$, $V \geq 0$ be a rank-one operator. Then 0 is an eigenvalue of $T - V$ if and only if*

$$1 = \text{tr} \left(V^{1/2} T^{-1} V^{1/2} \right)$$

Recall that

$$\ell P_{\dagger}^* \mathcal{Q}(\ell, \omega) P_{\dagger} = P_{\dagger}^* \mathcal{Q}_0 P_{\dagger} - \frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle \cdot, \Phi_{\ell}^{(\dagger)} \rangle_{\Gamma} \cdot \Phi_{\ell}^{(\dagger)} + \ell P_{\dagger}^* R(\ell, \omega) P_{\dagger}$$

for $\dagger \in \{\text{s, as}\}$. Note that multiplication with ℓ does not change the kernel of the corresponding operator. To deduce the asymptotics of the eigenvalue we apply the Birman-Schwinger principle with $H = L_{2, \dagger}(\Gamma)$,

$$T := \mathcal{Q}_0 + \ell R(\ell, \omega) \quad \text{and} \quad V := \frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle \cdot, \Phi_{\ell}^{(\dagger)} \rangle_{\Gamma} \cdot \Phi_{\ell}^{(\dagger)}.$$

Then

$$V^{1/2} = \sqrt{\frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}}} \cdot \frac{1}{\|\Phi_{\ell}^{(\dagger)}\|_{L_2(\Gamma)}} \langle \cdot, \Phi_{\ell}^{(\dagger)} \rangle_{\Gamma} \Phi_{\ell}^{(\dagger)}.$$

Let us now consider the symmetric case. For the choice $\omega = \lambda_1(\ell)$ the Birman-Schwinger principle implies

$$\frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{\Lambda - \lambda_1(\ell)} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle (\mathcal{Q}_0 + \ell R(\ell, \lambda_1(\ell)))^{-1} \Phi_{\ell}^{(\text{s})}, \Phi_{\ell}^{(\text{s})} \rangle_{\Gamma} = 1$$

or equivalently

$$\sqrt{\Lambda - \lambda_1(\ell)} = \frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{2\zeta_1''(\varkappa)}} \langle (\mathcal{Q}_0 + \ell R(\ell, \lambda_1(\ell)))^{-1} \Phi_{\ell}^{(\text{s})}, \Phi_{\ell}^{(\text{s})} \rangle_{\Gamma}.$$

Note that

$$\begin{aligned} (\mathcal{Q}_0 + \ell R(\ell, \omega))^{-1} &= (I + \ell \mathcal{Q}_0^{-1} R(\ell, \omega))^{-1} \mathcal{Q}_0^{-1} \\ &= \sum_{k=0}^{\infty} \ell^k (-\mathcal{Q}_0^{-1} R(\ell, \omega))^k \mathcal{Q}_0^{-1} = \mathcal{Q}_0^{-1} + \mathcal{O}(\ell), \end{aligned} \quad (5.41)$$

where the last estimate holds uniformly in $\omega \in (\Lambda, -\varepsilon, \Lambda)$. For sufficiently small ℓ the sum converges absolutely in $\mathcal{L}(L_2(\Gamma))$. Using the Taylor expansion of $\Phi_{\ell}^{(\text{s})}$ we obtain

$$\Phi_{\ell}^{(\text{s})}(x) = \cos(\varkappa \ell x) = 1 + \mathcal{O}(\ell) = \Psi_{\text{ct}}(x) + \mathcal{O}(\ell),$$

where $\Psi_{\text{ct}} = 1 \in L_{2, \text{s}}(\Gamma)$ is the constant function. Thus,

$$\begin{aligned} \sqrt{\Lambda - \lambda_1(\ell)} &= \frac{8\ell^2 \cdot |\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{2\zeta_1''(\varkappa)}} \langle (\mathcal{Q}_0 + \ell R(\ell, \lambda_1(\ell)))^{-1} \Phi_{\ell}^{(\text{s})}, \Phi_{\ell}^{(\text{s})} \rangle_{\Gamma} \\ &= \frac{8|\partial_2 \psi_{\varkappa, 2}(0)|^2}{\sqrt{2\zeta_1''(\varkappa)}} \cdot \langle \mathcal{Q}_0^{-1} \Psi_{\text{ct}}, \Psi_{\text{ct}} \rangle_{\Gamma} \cdot \ell^2 + \mathcal{O}(\ell^3). \end{aligned}$$

Setting

$$\nu_1 := \frac{32|\partial_2 \psi_{\varkappa, 2}(0)|^4}{\zeta_1''(\varkappa)} \cdot \langle \mathcal{Q}_0^{-1} \Psi_{\text{ct}}, \Psi_{\text{ct}} \rangle_{\Gamma}^2 \quad (5.42)$$

we obtain

$$\Lambda - \lambda_1(\ell) = \nu_1 \cdot \ell^4 + \mathcal{O}(\ell^5).$$

It remains to prove that $\nu_1 > 0$. Note that $\zeta_1''(\varkappa) > 0$ and

$$\langle \mathcal{Q}_0^{-1} \Psi_{\text{ct}}, \Psi_{\text{ct}} \rangle_{\Gamma} = \langle \mathcal{Q}_0^{-1/2} \Psi_{\text{ct}}, \mathcal{Q}_0^{-1/2} \Psi_{\text{ct}} \rangle_{\Gamma} > 0.$$

Moreover, using the explicit representation of the eigenfunction ψ_{\varkappa} in (4.11) we obtain

$$\partial_2 \psi_{\varkappa,2}(0) = c_1 \varkappa \left[\frac{\Lambda}{2} - \varkappa^2 \right] \sqrt{\Lambda - \varkappa^2} \left[\cos \left(\frac{\pi}{2} \sqrt{\frac{\Lambda}{2} - \varkappa^2} \right) - \cos \left(\frac{\pi}{2} \sqrt{\Lambda - \varkappa^2} \right) \right],$$

where $c_1 \neq 0$ is a normalising factor. A numerical calculation, which can be made rigorous by inserting the corresponding power series, shows that

$$\partial_2 \psi_{\varkappa,2}(0) \neq 0.$$

This proves the asymptotic formula for the eigenvalue $\lambda_1(\ell)$. The second eigenvalue is treated in the same way. Here we use the estimate

$$\Phi_{\ell}^{(\text{as})}(x) = \sin(\varkappa \ell x) = \varkappa \ell \cdot x + \mathcal{O}(\ell^2) = \varkappa \ell \cdot \Psi_{\text{id}}(x) + \mathcal{O}(\ell^2),$$

where $\Psi_{\text{id}}(x) = x$ is the identity function on Γ . As above we obtain

$$\Lambda - \lambda_2(\ell) = \nu_2 \cdot \ell^8 + \mathcal{O}(\ell^5)$$

where

$$\nu_2 := \frac{32 \cdot \varkappa^4 \cdot |\partial_2 \psi_{\varkappa,2}(0)|^4}{\zeta_1''(\varkappa)} \cdot \langle \mathcal{Q}_0^{-1} \Psi_{\text{id}}, \Psi_{\text{id}} \rangle_{\Gamma}^2 > 0. \quad (5.43)$$

For the sake of completeness we want to calculate the expressions $\langle \mathcal{Q}_0^{-1} \Psi_{\text{ct}}, \Psi_{\text{ct}} \rangle_{\Gamma}$ and $\langle \mathcal{Q}_0^{-1} \Psi_{\text{id}}, \Psi_{\text{id}} \rangle_{\Gamma}$ in the case of $\Gamma = (-1, 1)$. Then the operator \mathcal{Q}_0 becomes the composition of the standard finite Hilbert transform and the derivative. Using [8, Formula (4.8)] or [7, Section 5.2] we obtain

$$(\mathcal{Q}_0^{-1} \Psi_{\text{ct}})(x) = \sqrt{1 - x^2},$$

which implies

$$\langle \mathcal{Q}_0^{-1} \Psi_{\text{ct}}, \Psi_{\text{ct}} \rangle_{(-1,1)} = \int_{-1}^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{2}.$$

Moreover, using [8, Formula (4.9)] we obtain

$$(\mathcal{Q}_0^{-1} \Psi_{\text{id}})(x) = \frac{x}{2} \sqrt{1 - x^2},$$

and thus,

$$\langle \mathcal{Q}_0^{-1} \Psi_{\text{id}}, \Psi_{\text{id}} \rangle_{(-1,1)} = \frac{1}{2} \int_{-1}^1 x^2 \sqrt{1 - x^2} \, dt = \frac{\pi}{16}.$$

Thus, we obtain

$$\Lambda - \lambda_1(\ell) = \frac{8\pi^2 |\partial_2 \psi_{\varkappa,2}(0)|^4}{\zeta_1''(\varkappa)} \cdot \ell^4 + \mathcal{O}(\ell^5), \quad (5.44)$$

$$\Lambda - \lambda_2(\ell) = \frac{\pi^2 \varkappa^4 |\partial_2 \psi_{\varkappa,2}(0)|^4}{8\zeta_1''(\varkappa)} \cdot \ell^8 + \mathcal{O}(\ell^9). \quad (5.45)$$

This completes the proof of Theorem 3.1.

6. PROOF OF THE MAIN RESULT - 3D

We recall that in three dimensions we consider only circular cracks, since we want to take advantage of the rotational symmetry of the problem. We put $\Gamma := B(0, 1)$, $\Gamma_\ell := B(0, \ell)$ and consider the elasticity operator on $(\mathbb{R}^2 \times I) \setminus (\overline{\Gamma_\ell} \times \{0\})$ with traction-free boundary conditions. As in the two-dimensional case we reduce the original problem to a problem on the upper half-plate. This is done in exactly the same way, so we shall omit the details. Then for $\omega \in \mathbb{C}$ the corresponding Poisson problem on the upper half-plate reads as

$$(-\Delta - \text{grad div})u = \omega u \quad \text{in } \mathbb{R}^2 \times I_+, \quad (6.1)$$

and

$$\begin{cases} \partial_3 u_1 + \partial_1 u_3 = 0 \\ \partial_3 u_2 + \partial_2 u_3 = 0 \\ -2\partial_3 u_3 = 0 \end{cases} \quad \text{on } \mathbb{R}^2 \times \left\{ \frac{\pi}{2} \right\}, \quad (6.2)$$

$$\begin{cases} \partial_3 u_1 + \partial_1 u_3 = 0 \\ \partial_3 u_2 + \partial_2 u_3 = 0 \\ u_3 = g \end{cases} \quad \text{on } \mathbb{R}^2 \times \{0\}, \quad (6.3)$$

where $g \in H^{1/2}(\mathbb{R}^2)$ and $u \in H^1(\Omega_+; \mathbb{C}^3) \cap \mathcal{H}_{2+}$. The spaces \mathcal{H}_{1+} and \mathcal{H}_{2+} will not be separately introduced in the three-dimensional case since their definition is obvious.

Applying the Fourier transform with respect to the first two variables leads to the system

$$\begin{pmatrix} 2\xi_1^2 + \xi_2^2 - \partial_3^2 & \xi_1 \xi_2 & -i\xi_1 \partial_3 \\ \xi_1 \xi_2 & \xi_1^2 + 2\xi_2^2 - \partial_3^2 & -i\xi_2 \partial_3 \\ -i\xi_1 \partial_3 & -i\xi_2 \partial_3 & \xi_1^2 + \xi_2^2 - 2\partial_3^2 \end{pmatrix} \hat{u}(\xi, x_3) = \omega \hat{u}(\xi, x_3), \quad (6.4)$$

where $(\xi, x_3) \in \mathbb{R}^2 \times I_+$. Moreover, we have

$$\begin{cases} \partial_3 \hat{u}_1(\xi, \frac{\pi}{2}) + i\xi_1 \hat{u}_3(\xi, \frac{\pi}{2}) = 0 \\ \partial_3 \hat{u}_2(\xi, \frac{\pi}{2}) + i\xi_2 \hat{u}_3(\xi, \frac{\pi}{2}) = 0 \\ 2\partial_3 \hat{u}_3(\xi, \frac{\pi}{2}) = 0 \end{cases} \quad \text{for } \xi \in \mathbb{R}^2, \quad (6.5)$$

$$\begin{cases} \partial_3 \hat{u}_1(\xi, 0) + i\xi_1 \hat{u}_3(\xi, 0) = 0 \\ \partial_3 \hat{u}_2(\xi, 0) + i\xi_2 \hat{u}_3(\xi, 0) = 0 \\ \hat{u}_3(\xi, 0) = \hat{g}(\xi) \end{cases} \quad \text{for } \xi \in \mathbb{R}^2. \quad (6.6)$$

If $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ then the Sturm-Liouville problem (6.4)-(6.6) is uniquely solvable for any $\xi \in \mathbb{R}^2$ and we denote by $\hat{u}(\xi) = K_\omega(\xi)\hat{g}(\xi)$ its unique solution. The rotational

symmetry of the problem implies

$$K_\omega(M\xi) = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} K_\omega(\xi) \quad (6.7)$$

for every $M \in \text{SO}(2)$, cf. Lemma 4.3. In particular, we have to solve the corresponding Poisson problem only for $\xi = (|\xi|, 0)$. For $\omega \in \mathbb{C} \setminus [\Lambda, \infty)$ we denote by $K_\omega : H^{1/2}(\mathbb{R}^2) \rightarrow H^1(\Omega_+; \mathbb{C}^3) \cap \mathcal{H}_{2+}$ the Poisson operator and by $D_\omega : H^{1/2}(\mathbb{R}^2) \rightarrow H^{-1/2}(\mathbb{R}^2)$ the Dirichlet-to-Neumann operator. Then

$$(\widehat{K_\omega g})(\xi, \cdot) = K_\omega(\xi) \hat{g}(\xi).$$

If m_ω is given as in previous section, then a short calculation shows that the Dirichlet-to-Neumann operator satisfies

$$\widehat{D_\omega g}(\xi) = m_\omega(|\xi|) \hat{g}(\xi). \quad (6.8)$$

As in Lemma 5.2 we have the following mapping properties

$$K_\omega : H^s(\mathbb{R}^2) \rightarrow H^{s+1/2}(\Omega_+; \mathbb{C}^3), \quad D_\omega : H^s(\mathbb{R}^2) \rightarrow H^{s-1}(\mathbb{R}^2)$$

for all $s \in \mathbb{R}$. The remaining steps of the proof are well known. We define the spaces $\tilde{H}_0^{1/2}(\Gamma_\ell)$ and $H^{-1/2}(\Gamma_\ell)$ as in (5.19) and (5.20) and put

$$D_{\Gamma_\ell, \omega} : \tilde{H}_0^{1/2}(\Gamma_\ell) \rightarrow H^{-1/2}(\Gamma_\ell), \quad D_{\Gamma_\ell, \omega} := r_{\Gamma_\ell} D_{\Gamma_\ell, \omega} e_{\Gamma_\ell}, \quad (6.9)$$

where e_{Γ_ℓ} is the extension operator and r_{Γ_ℓ} is the restriction operator. Let

$$d_{\Gamma_\ell, \omega}[g, h] := \int_{\mathbb{R}^2} m_\omega(|\xi|) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi, \quad g, h \in D[d_{\Gamma_\ell, \omega}] := \tilde{H}_0^{1/2}(\Gamma_\ell)$$

be the associated sesquilinear form and define the scaled operator

$$\mathcal{Q}(\ell, \omega) : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma), \quad \mathcal{Q}(\ell, \omega) = \ell \cdot T_\ell^* D_{\Gamma_\ell, \omega} T_\ell \quad (6.10)$$

as well as its sesquilinear form

$$q(\ell, \omega)[g, h] = \int_{\mathbb{R}^2} m_\omega(|\xi|/\ell) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi, \quad g, h \in D[q(\ell, \omega)] := \tilde{H}_0^{1/2}(\Gamma). \quad (6.11)$$

Here $T_\ell : L_2(\Gamma) \rightarrow L_2(\Gamma_\ell)$, $(T_\ell g)(\hat{x}) := \ell^{-1} g(\hat{x}/\ell)$. As before we define $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$,

$$\langle \mathcal{Q}_0 g, h \rangle_\Gamma := q_0[g, h] := \int_{\mathbb{R}^2} |\xi| \cdot \hat{g}(\xi) \cdot \overline{\hat{h}(\xi)} \, d\xi. \quad (6.12)$$

Using a three-dimensional version of Theorem 5.5 we obtain

$$\mathcal{Q}(\ell, \omega) = \mathcal{Q}(\ell, 0) - \omega \cdot T_\ell^* K_0^* (I + \omega(A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 T_\ell,$$

or equivalently

$$q(\ell, \omega)[g, h] = q(\ell, 0)[g, h] - \omega \langle (I + \omega(A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 T_\ell g, K_0 T_\ell h \rangle_{\Omega_+}.$$

Then the estimate $m_0(|\xi|) = |\xi| + \mathcal{O}(1)$ directly implies that

$$q(\ell, 0)[g, h] = \frac{1}{\ell} q_0[g, h] + \mathcal{O}(1).$$

Next we give an estimate for the resolvent term. For $\theta \in \mathbb{R}^2$, $|\theta| = 1$, we define

$$\Phi_{\varkappa\theta}(\hat{x}) := e^{i\varkappa\theta \cdot \hat{x}}, \quad \hat{x} \in \mathbb{R}^2,$$

and we denote by $P_{\varkappa\theta}$ the projection in $L_2(\Gamma)$ on the subspace spanned by the function $\Phi_{\varkappa\theta}$. Moreover, let $\psi_{\varkappa\theta} = (\psi_{\varkappa\theta,1}, \psi_{\varkappa\theta,2}, \psi_{\varkappa\theta,3})^T$ be chosen such that $\|\psi_{\varkappa\theta}\|_{L_2(I_+; \mathbb{C}^3)} = 1$ and

$$A_{\varnothing+}^{(2)}(\varkappa\theta)\psi_{\varkappa\theta} = \Lambda\psi_{\varkappa\theta}.$$

Theorem 6.1. *There exists $\ell_0 > 0$ and $\varepsilon > 0$ such that for all $\ell \in (0, \ell_0)$ and $|\omega - \Lambda| < \varepsilon$ the following expansion holds true*

$$\mathcal{Q}(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 - \frac{2\varkappa \cdot |\partial_3 \psi_{(\varkappa,0),3}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \cdot \int_{\{|\theta|=1\}} T_\ell^* P_{\varkappa\theta} T_\ell \, d\theta + R(\ell, \omega). \quad (6.13)$$

The remainder satisfies the estimate

$$\sup\{\|R(\ell, \omega)\|_{\mathcal{L}(L_2(\Gamma))} : \ell \in (0, \ell_0) \wedge |\omega - \Lambda| < \varepsilon\} < \infty.$$

Proof. We have

$$\begin{aligned} \mathcal{Q}(\ell, \omega) &= \frac{1}{\ell} \mathcal{Q}_0 - \omega \cdot T_\ell^* K_0^* (I + (A_{\varnothing+}^{(2)} - \omega)^{-1}) K_0 T_\ell + \mathcal{O}(1) \\ &= \frac{1}{\ell} \mathcal{Q}_0 - \omega \cdot T_\ell^* K_0^* (A_{\varnothing+}^{(2)} - \omega)^{-1} K_0 T_\ell + \mathcal{O}(1). \end{aligned}$$

For $g, h \in \tilde{H}^{1/2}(\Gamma)$ we put $g_\ell = T_\ell g$ and $h_\ell = T_\ell h$. Then

$$\langle T_\ell^* K_0^* (A_{\varnothing+}^{(2)} - \omega)^{-1} K_0 T_\ell g, h \rangle_{\Omega_+} = \int_{\mathbb{R}^2} \langle (A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1} K_0(\xi) \hat{g}_\ell(\xi), K_0(\xi) \hat{h}_\ell(\xi) \rangle_{I_+} \, d\xi.$$

Introducing polar coordinates and using an estimate on the resolvent term we obtain as in the two-dimensional case

$$\begin{aligned} &\langle T_\ell^* K_0^* (A_{\varnothing+}^{(2)} - \omega)^{-1} K_0 T_\ell g, h \rangle_{\Omega_+} \\ &= \int_{\varkappa - \varepsilon < |\xi| < \varkappa + \varepsilon} \langle (A_{\varnothing+}^{(2)}(\xi) - \omega)^{-1} K_0(\xi) \hat{g}_\ell(\xi), K_0(\xi) \hat{h}_\ell(\xi) \rangle_{I_+} \, d\xi + \mathcal{O}(1) \\ &= \int_{\{\theta=1\}} \int_{\varkappa - \varepsilon}^{\varkappa + \varepsilon} \langle (A_{\varnothing+}^{(2)}(\theta r) - \omega)^{-1} K_0(\theta r) \hat{g}_\ell(\theta r), K_0(\theta r) \hat{h}_\ell(\theta r) \rangle_{I_+} r \, dr \, d\theta + \mathcal{O}(1) \\ &= \int_{\{\theta=1\}} \int_{\varkappa - \varepsilon}^{\varkappa + \varepsilon} \frac{1}{\zeta_1(r) - \omega} \langle P_1(\theta r) K_0(\theta r) \hat{g}_\ell(\theta r), K_0(\theta r) \hat{h}_\ell(\theta r) \rangle_{I_+} r \, dr \, d\theta + \mathcal{O}(1). \end{aligned}$$

We recall that the eigenvalue distribution functions $\zeta_k(\cdot)$, $k \in \mathbb{N}$, are given as in the two-dimensional case. The operator $P_1(\cdot)$ is the projection onto the corresponding eigenspace. Using the residue theorem for the inner integral and a Taylor expansion of the remaining terms we obtain

$$\begin{aligned} &\int_{\varkappa - \varepsilon}^{\varkappa + \varepsilon} \frac{1}{\zeta_1(r) - \omega} \langle P_1(\theta r) K_0(\theta r) \hat{g}_\ell(\theta r), K_0(\theta r) \hat{h}_\ell(\theta r) \rangle_{I_+} \cdot r \, dr \\ &= \frac{2\pi\varkappa}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle P_1(\theta\varkappa) K_0(\theta\varkappa) \hat{g}_\ell(\theta\varkappa), K_0(\theta\varkappa) \hat{h}_\ell(\theta\varkappa) \rangle_{I_+} + \mathcal{O}(1), \end{aligned}$$

where the reminder may be estimated uniformly in $|\theta| = 1$. Then

$$\begin{aligned} & \Lambda^2 \langle P_1(\theta \varkappa) K_0(\theta \varkappa) \hat{g}_\ell(\theta \varkappa), K_0(\theta \varkappa) \hat{h}_\ell(\theta \varkappa) \rangle_{I_+} \\ &= \Lambda^2 \langle K_0(\varkappa) g_\ell(\varkappa \theta), \psi_{\varkappa \theta} \rangle_{I_+} \cdot \langle \psi_{\varkappa \theta}, K_0(\varkappa \theta) \hat{h}_\ell(\varkappa \theta) \rangle_{I_+} \\ &= |2\partial_3 \psi_{\theta \varkappa, 3}(0)|^2 \cdot \hat{g}_\ell(\theta \varkappa) \cdot \overline{\hat{h}_\ell(\theta \varkappa)}. \end{aligned}$$

We note that $\psi_{\varkappa \theta, 3} = \psi_{(\varkappa, 0), 3}$ does not depend on θ . Moreover, from the particular choice $\Gamma = B(0, 1)$ we obtain

$$\begin{aligned} \hat{g}_\ell(\theta \varkappa) \cdot \overline{\hat{h}_\ell(\theta \varkappa)} &= \frac{1}{4\pi^2} \left(\int_\Gamma e^{-i\varkappa \theta x} g_\ell(x) \, dx \right) \overline{\left(\int_\Gamma e^{-i\varkappa \theta x} h_\ell(x) \, dx \right)} \\ &= \frac{1}{4\pi} \langle P_{\varkappa \theta} T_\ell g, T_\ell h \rangle_\Gamma = \frac{1}{4\pi} \langle T_\ell^* P_{\varkappa \theta} T_\ell g, h \rangle_\Gamma. \end{aligned}$$

This proves the assertion. \square

Next we want to use the rotational symmetry of the problem. Recall that $\Gamma = B(0, 1)$. For $m \in \mathbb{Z}$ we introduce the space

$$L_{2,m}(\Gamma) := \{g \in L_2(\Gamma) : g(r \cos \varphi, r \sin \varphi) = e^{im\varphi} \tilde{g}(r) \text{ for some } \tilde{g} : (0, 1) \rightarrow \mathbb{C}\} \quad (6.14)$$

with the corresponding projection

$$(P_m g)(r \cos \varphi, r \sin \varphi) = \frac{1}{2\pi} e^{im\varphi} \int_0^{2\pi} e^{-imt} g(r \cos t, r \sin t) \, dt, \quad (6.15)$$

where $(r, \varphi) \in (0, 1) \times (0, 2\pi)$. Then

$$L_2(\Gamma) = \bigoplus_{m \in \mathbb{Z}} L_{2,m}(\Gamma).$$

Lemma 6.2. *The form $q(\ell, \omega)$ admits the following decomposition*

$$q(\ell, \omega) := \bigoplus_{m \in \mathbb{Z}} q^{(m)}(\ell, \omega),$$

where $q^{(m)}(\ell, \omega)$ acts in the Hilbert spaces $L_{2,m}(\Gamma)$.

Proof. For the proof we use a similar decomposition of the L_2 -space of functions defined on all of \mathbb{R}^2 . We have

$$L_2(\mathbb{R}^2) = \bigoplus_{m \in \mathbb{Z}} L_{2,m}(\mathbb{R}^2),$$

where

$$L_{2,m}(\mathbb{R}^2) = \{g \in L_2(\mathbb{R}^2) : g(r \cos \varphi, r \sin \varphi) = e^{im\varphi} \tilde{g}(r) \text{ for some } \tilde{g} : \mathbb{R}_+ \rightarrow \mathbb{C}\}.$$

Let us denote by \tilde{P}_m the corresponding projections. A short calculation shows that \tilde{P}_m commutes with the standard Fourier transform on \mathbb{R}^2 . For $g \in H^{1/2}(\mathbb{R}^2)$ we have

$$\begin{aligned} \|P_m g\|_{H^{1/2}(\mathbb{R}^2)}^2 &= \|(1 + |\xi|)^{1/2} \widehat{(\tilde{P}_m g)}(\xi)\|_{L_2(\mathbb{R}_\xi^2)}^2 = \|(1 + |\xi|)^{1/2} (\tilde{P}_m \hat{g})(\xi)\|_{L_2(\mathbb{R}_\xi^2)}^2 \\ &= \|\tilde{P}_m((1 + |\xi|)^{1/2} g(\xi))\|_{L_2(\mathbb{R}_\xi^2)}^2 \leq \|g\|_{H^{1/2}(\mathbb{R}^2)}^2, \end{aligned}$$

and thus, $g \in H^{1/2}(\mathbb{R}^2)$. In the same way we obtain for $g, h \in H^{1/2}(\mathbb{R})$

$$\langle \tilde{P}_m g, h \rangle_{H^{1/2}(\mathbb{R})} = \langle g, \tilde{P}_m h \rangle_{H^{1/2}(\mathbb{R})} \quad \text{and} \quad \langle \tilde{P}_{m_1} g, \tilde{P}_{m_2} h \rangle_{H^{1/2}(\mathbb{R})} = 0 \quad \text{if } m_1 \neq m_2.$$

Thus, \tilde{P}_m is a orthogonal projection in $H^{1/2}(\mathbb{R}^2)$ and

$$H^{1/2}(\mathbb{R}^2) = \bigoplus_{m \in \mathbb{Z}} H^{1/2}(\mathbb{R}^2) \cap L_{2,m}(\mathbb{R}^2).$$

A similar assertion holds true for $\tilde{H}_0^{1/2}(\Gamma)$ equipped with the standard scalar product. Now consider $\tilde{H}_0^{1/2}(\Gamma)$ with the scalar product induced by the form $q(\ell, \omega)$. Note that for $m_1 \neq m_2$ we have

$$q(\ell, \omega)[P_{m_1} g, P_{m_2} h] = \int_{\mathbb{R}^2} m_\omega(|\xi|/\ell) \cdot (\tilde{P}_{m_1} \hat{g})(\xi) \overline{(\tilde{P}_{m_2} \hat{h})(\xi)} \, d\xi = 0.$$

As above we obtain

$$D[q(\ell, \omega)] = \bigoplus_{m \in \mathbb{Z}} D[q(\ell, \omega)] \cap L_{2,m}(\Gamma),$$

which proves the assertion. \square

Remark. Following [1] we have the decomposition

$$\mathcal{H}_{2+} = \bigoplus_{m \in \mathbb{Z}} X_m,$$

where

$$X_m := \left\{ u \in L^2(\Omega_+; \mathbb{C}^3) \cap \mathcal{H}_{2+} : u(r \cos \varphi, r \sin \varphi, x_3) = e^{im\varphi} \begin{pmatrix} M_\varphi & 0 \\ 0 & 1 \end{pmatrix} \tilde{u}(r, x_3) \right. \\ \left. \text{for some } \tilde{u} : \mathbb{R}_+ \times I_+ \rightarrow \mathbb{C} \right\},$$

Here we denote by $M_\varphi \in \text{SO}(2)$ the planar rotation matrix to the rotation angle $\varphi \in (0, 2\pi)$. The elasticity operator with circular crack decomposes as follows

$$A_{\Gamma_{\ell+}}^{(2)} = \bigoplus_{m \in \mathbb{Z}} A_{\Gamma_{\ell+}}^{(2),m},$$

where $A_{\Gamma_{\ell+}}^{(2),m}$ acts in X_m . A short calculations shows that

$$\dim \ker \left(\mathcal{Q}_0(\ell, \omega) \Big|_{\tilde{H}_0^{1/2}(\Gamma) \cap L_{2,m}(\Gamma)} \right) = \dim \ker \left(A_{\Gamma_{\ell+}}^{(2),m} - \omega \right).$$

As a consequence of Lemma 6.2 we obtain

$$\mathcal{Q}(\ell, \omega) = \sum_{m \in \mathbb{Z}} P_m^* \mathcal{Q}(\ell, \omega) P_m,$$

where

$$P_m^* \mathcal{Q}(\ell, \omega) P_m \\ = \frac{1}{\ell} P_m^* \mathcal{Q}_0 P_m - \frac{2\kappa \cdot |\partial_3 \psi_{\kappa(1,0),3}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\kappa)}} \cdot \int_{\{|\theta|=1\}} P_m^* T_\ell^* P_{\kappa\theta} T_\ell P_m \, d\theta + P_m^* R(\ell, \omega) P_m.$$

Fix $\theta = (\cos \alpha, \sin \alpha)^T$, $\alpha \in (0, 2\pi)$. Then for $g, h \in L_2(\Gamma)$ we obtain

$$\langle P_m^* T_\ell^* P_{\varkappa\theta} T_\ell P_m g, h \rangle_\Gamma = \frac{1}{\pi} \langle g, P_m T_\ell^* \Phi_{\varkappa\theta} \rangle_\Gamma \cdot \langle P_m T_\ell^* \Phi_{\varkappa\theta}, h \rangle_\Gamma,$$

where we recall that $\Phi_{\varkappa\theta}(\hat{x}) := e^{i\varkappa\theta \cdot \hat{x}}$, $\hat{x} \in \mathbb{R}^2$. For $(r, \phi) \in (0, 1) \times (0, 2\pi)$ we have

$$\begin{aligned} (P_m T_\ell^* \Phi_{\varkappa\theta})(r \cos \varphi, r \sin \varphi) &= \frac{\ell e^{im\varphi}}{2\pi} \int_0^{2\pi} e^{-im\psi} e^{i\varkappa\ell(\cos \alpha, \sin \alpha) \cdot (r \cos \psi, r \sin \psi)^T} d\psi \\ &= \ell \cdot e^{im\varphi} \cdot e^{-im\alpha} \cdot \frac{1}{2\pi} \int_{\alpha-\frac{\pi}{2}}^{\alpha-\frac{5\pi}{2}} e^{-ir\varkappa\ell \sin t + im(t+\frac{\pi}{2})} dt \\ &= \ell \cdot i^m \cdot e^{-im\alpha} \cdot e^{im\varphi} \cdot J_m(r\varkappa\ell), \end{aligned}$$

where J_m denotes the Bessel function of the first kind of order m , cf. [21, Formula 2.2 (5)]. Setting

$$\Phi_\ell^{(m)}(r \cos \varphi, r \sin \varphi) := e^{im\varphi} J_m(r\varkappa\ell)$$

we obtain

$$P_m^* T_\ell^* P_{\varkappa\theta} T_\ell P_m = \frac{\ell^2}{\pi} \langle \cdot, \Phi_\ell^{(m)} \rangle_\Gamma \Phi_\ell^{(m)}.$$

In particular this expression does not depend on θ any more. Thus, we have

$$P_m^* \mathcal{Q}(\ell, \omega) P_m = \frac{1}{\ell} P_m^* \mathcal{Q}_0 P_m - \frac{4\varkappa \cdot \ell^2 \cdot |\partial_3 \psi_{\varkappa(1,0),3}(0)|^2}{\sqrt{\Lambda - \omega} \cdot \sqrt{2\zeta_1''(\varkappa)}} \langle \cdot, \Phi_\ell^{(m)} \rangle_\Gamma \Phi_\ell^{(m)} + P_m^* R(\ell, \omega) P_m.$$

Note that now the singular term is again a rank-one perturbation. As above we may prove that for every $m \in \mathbb{Z}$ there exists $\ell_0 = \ell_0(m)$ such that the operator $A_{\Gamma_\ell^+}^{(2),m}$ has a unique eigenvalue $\lambda_m(\ell)$ below its essential spectrum $[\Lambda, \infty)$. Applying the Birman-Schwinger principle we obtain for the eigenvalue

$$\begin{aligned} \sqrt{\Lambda - \lambda_m(\ell)} &= \ell^3 \cdot \frac{4\varkappa \cdot |\partial_3 \psi_{\varkappa(1,0),3}(0)|^2}{\sqrt{2\zeta_1''(\varkappa)}} \cdot \langle (\mathcal{Q}_0 + \ell R_{\ell, \lambda_m(\ell)})^{-1} \Phi_\ell^{(m)}, \Phi_\ell^{(m)} \rangle_\Gamma \\ &= \ell^3 \cdot \frac{4\varkappa \cdot |\partial_3 \psi_{\varkappa(1,0),3}(0)|^2}{\sqrt{2\zeta_1''(\varkappa)}} \cdot \langle \mathcal{Q}_0^{-1} \Phi_\ell^{(m)}, \Phi_\ell^{(m)} \rangle_\Gamma + \mathcal{O}(\ell^4). \end{aligned}$$

For the function $\Phi_\ell^{(m)}$ we use its Taylor expansion in the radial direction. For $m \geq 0$ we have

$$\begin{aligned} \Phi_\ell^{(m)}(r, \varphi) &= e^{im\varphi} J_m(r\varkappa\ell) = \frac{\ell^m \varkappa^m}{2^m} r^m e^{im\varphi} + \mathcal{O}(\ell^{m+1}) \\ &= \frac{\ell^m \varkappa^m}{2^m} \cdot \Psi_m(r, \varphi) + \mathcal{O}(\ell^{m+1}), \end{aligned}$$

where $\Psi_m(r, \varphi) = r^m \cdot e^{im\varphi}$. For $m \leq 0$ we have

$$\Phi_\ell^{(m)}(r, \varphi) = e^{im\varphi} (-1)^m J_{|m|}(r\varkappa\ell) = (-1)^m \frac{\ell^{|m|} \varkappa^{|m|}}{2^{|m|}} \cdot \Psi_m(r, \varphi) + \mathcal{O}(\ell^{|m|+1}),$$

where we put $\Psi_m(r, \phi) = r^{|m|} \cdot e^{im\phi}$. Finally, we have

$$\Lambda - \lambda_m(\ell, m) = \rho_m \cdot \ell^{6+4|m|} + \mathcal{O}(\ell^{7+4|m|}) \quad \text{as } \ell \rightarrow 0, \quad (6.16)$$

where

$$\rho_m := \ell^{6+4|m|} \cdot \frac{8\kappa^{4|m|+2} \cdot |\partial_3 \psi_{(\kappa,0),3}(0)|^4}{2^{4|m|} \cdot \zeta_1''(\kappa)} \cdot \langle \mathcal{Q}_0^{-1} \Psi_m, \Psi_m \rangle_\Gamma^2. \quad (6.17)$$

We note that $\rho_m > 0$, which follows from

$$\langle \mathcal{Q}_0^{-1} \Psi_m, \Psi_m \rangle_\Gamma = \langle \mathcal{Q}_0^{-1/2} \Psi_m, \mathcal{Q}_0^{-1/2} \Psi_m \rangle_\Gamma > 0$$

and from $\partial_3 \psi_{(\kappa,0),3}(0) \neq 0$. This proves Theorem 3.2.

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REFERENCES

- [1] Hänel, A. and Schulz, C. and Wirth, J., *Embedded eigenvalues for an elastic strip with cracks*. Quart. J. Mech. Appl. Math. **65**, (2012), 535–554.
- [2] Roitberg, I. and Vassiliev, D. and Weidl, T., *Edge resonance in an elastic semi-strip*. Quart. J. Mech. Appl. Math. **51**, (1998), 1–13.
- [3] Förster, C. and Weidl, T., *Trapped modes for an elastic strip with perturbation of the material properties*. Quart. J. Mech. Appl. Math. **59**, (2006), 399–418.
- [4] Förster, C., *Trapped modes for an elastic plate with a perturbation of Young's modulus*. Comm. Partial Differential Equations **33**, (2008), 1339–1367.
- [5] Förster, C. and Weidl, T., *Trapped modes in an elastic plate with a hole*. Algebra i Analiz **23**, (2011), 255–288 (in Russian).
- [6] Hänel, A. and Weidl, T., *Spectral asymptotics for the Dirichlet Laplacian with a Neumann window via a Birman-Schwinger analysis of the Dirichlet-to-Neumann operator*. arXiv:1511.05529.
- [7] Ammari, H. and Kang, H. and Lee, H., *Layer potential techniques in spectral analysis*. American Mathematical Society, Providence, RI, (2009).
- [8] Ammari, H. and Kang, H. and Lee, H. and Lim, J., *Boundary perturbations due to the presence of small linear cracks in an elastic body*. J. Elasticity **113**, (2013), 75–91.
- [9] Gridin, D. and Adamou, A. T. I. and Craster, R. V., *Trapped modes in curved elastic plates*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461**, (2005), 1181–1197.
- [10] Gridin, D. and Adamou, A. T. I. and Craster, R. V., *Trapped modes in bent elastic rods*. Wave Motion **42**, (2005), 352–366.
- [11] Zernov, V. and Pichugin, A. V. and Kaplunov, J., *Eigenvalue of a semi-infinite elastic strip*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **462**, (2006), 1255–1270.
- [12] Zernov, V. and Kaplunov, J., *Three-dimensional edge waves in plates*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **464**, (2008), 301–318.
- [13] Lawrie, J. B. and Kaplunov, J., *Edge waves and resonance on elastic structures: an overview*. Math. Mech. Solids **17**, (2012), 4–16.
- [14] Pagneux, V., *Complex resonance and localized vibrations at the edge of a semi-infinite elastic cylinder*. Math. Mech. Solids **17**, (2012), 17–26.
- [15] Birman, M. Š., *Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions*. Vestnik Leningrad. Univ. **17**, (1962), 22–55 (in Russian).
- [16] Kato, T., *Perturbation theory for linear operators*. Springer-Verlag, Berlin, (1995).
- [17] McLean, W., *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, (2000).
- [18] Kozlov, V. A. and Maz'ya, V. G. and Rossmann, J., *Elliptic boundary value problems in domains with point singularities*. American Mathematical Society, Providence, RI, (1997).

- [19] Behrndt, J. and Langer, M., *Boundary value problems for elliptic partial differential operators on bounded domains*. J. Funct. Anal. **243**, (2007), 536–565.
- [20] Gohberg, I. C. and Sigal, E. I., *An operator generalization of the logarithmic residue theorem and Rouché's theorem*. Mat. Sb. (N.S.) **84**, (1971) 607–629.
- [21] Watson, G. N., *A treatise on the theory of Bessel functions*. Cambridge University Press, Cambridge, (1995).

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